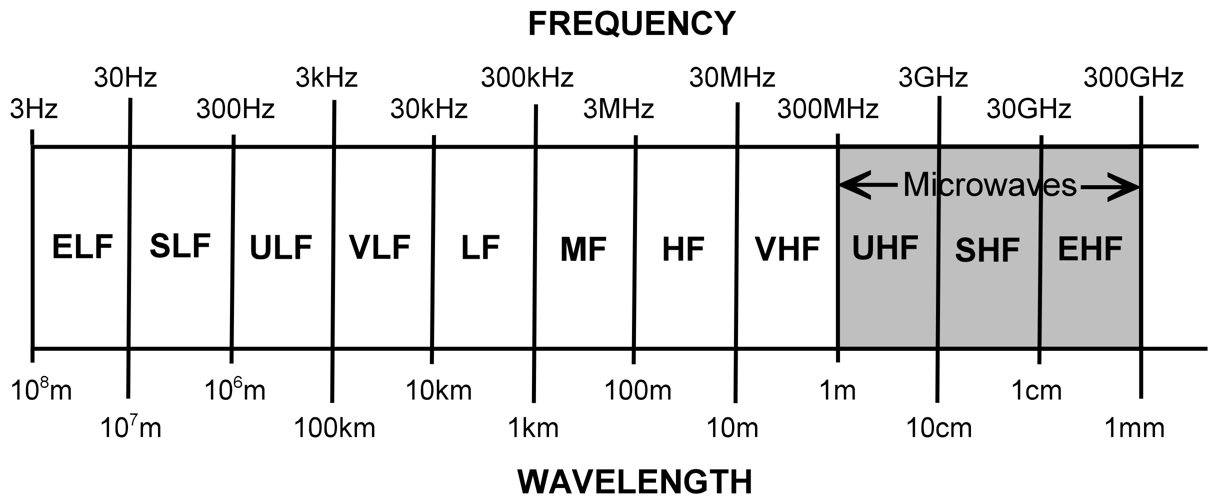
Microwaves

Microwaves in the Electromagnetic Spectrum (300 MHz - 300 GHz)



ELF	Extremely Low Frequency	3-30 Hz
SLF	Super Low Frequency	30-300 Hz
ULF	Ultra Low Frequency	300 Hz - 3 kHz
VLF	Very Low Frequency	3 kHz - 30 kHz
LF	Low Frequency	30 kHz - 300 kHz
MF	Medium Frequency	300 kHz - 3 MHz
HF	High Frequency	3 MHz - 30 MHz
VHF	Very High Frequency	30 MHz - 300 MHz
UHF	*Ultra High Frequency (decimeter waves)*	*300 MHz - 3 GHz*
SHF	*Super High Frequency (centimeter waves)*	*3 GHz - 30 GHz*
EHF	*Extremely High Frequency (millimeter waves)*	*30 GHz - 300 GHz*
?	(submillimeter waves)	300 GHz - 3000 GHz
IR	Infared	3000 GHz - 416,000 GHz

Microwave Properties

High bandwidth - The microwave frequency range (300 MHz - 300 GHz) is 999 times that of the entire frequency range below it.

Effect of the ionosphere - When lower frequency waves are directed upward into the atmosphere, they experience significant reflection due to the ionosphere. The lower frequency waves which pass through the ionosphere suffer distortion. Microwaves pass through the ionosphere with little effect and are therefore utilized in satellite communications and space transmissions.

Line-of-sight transmission/reception - The microwave receive antenna must be within the line-of-sight of the transmit antenna. Long distance communication on earth requires that microwave relay stations be used.

Electromagnetic noise characteristics - The electromagnetic noise level in nature over the 1-10 GHz frequency range is small. This allows for the detection of very low signal levels using sensitive receivers.

Antenna gain and directivity - The gain of an antenna is directly proportional to its electrical size. The beamwidth of an antenna is inversely proportional to the electrical size of its maximum dimension. Shorter wavelengths at microwave frequencies allow for smaller antennas. At higher frequencies (visible light-lasers), the beamwidth gets very small and pointing accuracy of the detector becomes a problem.

Target reflection of electromagnetic waves (radar cross section) - In general, electrically large conducting radar targets reflect more energy (shape is also a factor - stealth design). Thus, the higher frequencies of microwaves are preferred for radar systems. At millimeter waves, the wavelength becomes comparable to the size of raindrops which results in attenuation of the incident waves.

Absorption at resonant frequencies - various materials absorb microwave energy (dissipated in the form of heat) at specific resonant frequencies.

Applications of Microwaves

- Wireless communications
  - Personal Communications Systems (PCS)
    - (pagers, cell phones, etc.)
  - Global Positioning Satellite (GPS) Systems
  - Wireless Local Area Computer Networks (WLANS)
  - Direct Broadcast Satellite (DBS) Television
  - Telephone Microwave/Satellite Links, etc.

- Remote sensing
  - Radar (active remote sensing - radiate and receive)
    - Military applications (target tracking)
    - Weather radar
    - Ground Penetrating Radar (GPR)
    - Agricultural applications
  - Radiometry (passive remote sensing - receive inherent emissions)
    - Radio astronomy

- Industrial and home applications
  - Cooking, drying, heating
  - Microwave spectroscopy - molecular properties of materials can be determined by passing microwaves through a sample of the material and measuring the absorption spectrum.

Analysis Techniques in Microwave Theory

In general, circuit theory is not applicable to microwave problems. Circuit theory is derived from Maxwell's equations based on certain assumptions about the fields within the circuit elements. Specifically, the circuit elements must be small relative to wavelength for circuit equations to be valid. In this sense, microwave components must be modeled by distributed elements, not lumped elements. For this reason, we must use field theory solutions (Maxwell's equations) for microwave applications.

Maxwell's Equations

Maxwell's Equations (instantaneous, symmetric form)

$$\nabla \times \mathscr{E} = -\frac{\partial \mathscr{B}}{\partial t} - \mathscr{M} \qquad (\textit{Faraday's law})$$

$$\nabla \times \mathscr{H} = \frac{\partial \mathscr{D}}{\partial t} + \mathscr{J} \qquad (\textit{Ampere's law})$$

$$\nabla \cdot \mathscr{D} = \rho \qquad (\textit{Gauss' law - electric fields})$$

$$\nabla \cdot \mathscr{B} = \rho_m \qquad (\textit{Gauss' law - magnetic fields})$$

$\mathscr{E}, \mathscr{H}, \mathscr{D}, \mathscr{B}, \mathscr{J}, \mathscr{M}$ - instantaneous vectors [$\mathscr{E} = \mathscr{E}(x,y,z,t)$, etc.]
ρ, ρ_m - instantaneous scalars [$\rho = \rho(x,y,z,t)$, etc.]

$\mathscr{E}$ - electric field intensity (V/m)	$\mathscr{H}$ - magnetic field intensity (A/m)
$\mathscr{D}$ - electric flux density (C/m^2)	$\mathscr{B}$ - magnetic flux density (Wb/m^2)
$\mathscr{J}$ - electric current density (A/m^2)	$\mathscr{M}$ - magnetic current density (V/m^2)
ρ - electric charge density (C/m^3)	ρ_m - magnetic charge density (Wb/m^3)

The quantities of magnetic current density $\mathscr{M}$ and magnetic charge density ρ_m are nonphysical and included in the symmetric forms of Maxwell's equation for mathematical convenience. These magnetic sources may be used to simplify the mathematics of particular problems involving actual electric currents and charges.

The flux and field quantities are related by the constitutive relations:

$$\mathscr{D} = \varepsilon \mathscr{E} \qquad \mathscr{B} = \mu \mathscr{H}$$

where ε is the permittivity (F/m) and μ is the permeability (H/m) of the medium in which the fields are located. The permittivity and permeability of a given medium may be defined in terms of the free space (vacuum) values [$\varepsilon_o = 8.854\times10^{-12}$ F/m, $\mu_o = 4\pi\times10^{-7}$ H/m] and unitless relative values (μ_r, ε_r) such that

$$\varepsilon = \varepsilon_r \varepsilon_o \qquad \mu = \mu_r \mu_o$$

The instantaneous Maxwell's equations are valid given any type of time-dependence for the electromagnetic fields. Most applications in microwave engineering involve fields which have a sinusoidal (harmonic) time-dependence. This harmonic time-dependence allows us to simplify Maxwell's equations by writing them in terms of phasors just like we use in circuit analysis.

For time-harmonic fields, we may separate the dependence on time and space. The real-valued instantaneous electric field $\mathcal{E}(x,y,z,t)$ may be written as

$$\mathcal{E}(x,y,z,t) = \boldsymbol{E_o}(x,y,z)\cos(\omega t + \varphi)$$

$$= \underbrace{[E_o(x,y,z)\boldsymbol{a_E}]}_{\substack{\text{Real vector}\\ \text{[magnitude/direction]}}}\cos(\omega t + \varphi)$$

where $\boldsymbol{a}_E$ is a unit vector in the direction of the vector electric field. The arbitrary phase shift φ allows us to use the cosine function to represent any sinusoidal time variation relative to time t = 0. According to Euler's identity, we may write the equation above as

$$\mathcal{E}(x,y,z,t) = Re\left\{[E_o(x,y,z)\boldsymbol{a_E}]e^{j(\omega t + \varphi)}\right\}$$

$$= Re\left\{[E_o(x,y,z)e^{j\varphi}\boldsymbol{a_E}]e^{j\omega t}\right\}$$

$$= Re\left\{\underbrace{[\boldsymbol{E}(x,y,z)]}_{\substack{\text{Complex vector (phasor vector)}\\ \text{[magnitude/phase/direction]}}}e^{j\omega t}\right\}$$

We may write all vector quantities in the instantaneous Maxwell's equations in terms of phasors according to the relationship above. The derivatives with respect to time in the instantaneous equations yield $j\omega$ terms in the phasor equations.

Maxwell's Equations (phasor form)

$$\nabla \times \boldsymbol{E} = -j\omega \boldsymbol{B} - \boldsymbol{M} \qquad (Faraday's\ law)$$

$$\nabla \times \boldsymbol{H} = j\omega \boldsymbol{D} + \boldsymbol{J} \qquad (Ampere's\ law)$$

$$\nabla \cdot \boldsymbol{D} = \rho \qquad (Gauss'\ law\ -\ electric\ fields)$$

$$\nabla \cdot \boldsymbol{B} = \rho_m \qquad (Gauss'\ law\ -\ magnetic\ fields)$$

E, H, D, B, J, M - phasor vectors
ρ, ρ_m - phasor scalars

Relation of instantaneous quantities to phasor quantities ...

$$\mathscr{E}(x,y,z,t) = \mathrm{Re}\{\boldsymbol{E}(x,y,z)e^{j\omega t}\}, \text{ etc.}$$

Complex Permittivity and Permeability

In order to account for dielectric and magnetic losses in media where time-harmonic electromagnetic fields exist, we may define a complex permittivity and permeability.

$$\varepsilon = \varepsilon' - j\varepsilon''$$
$$\mu = \mu' - j\mu''$$

In the case of dielectrics, we may combine the conductivity losses with the dielectric losses according to Maxwell's equations. The conduction current density $\boldsymbol{J}$ in a given medium is defined by

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

where σ is the conductivity of the medium in S/m (℧/m). We may write a single equation which includes dielectric and conductor losses by incorporating the complex permittivity and the conduction current equation into Ampere's law.

$$
\begin{aligned}
\nabla \times \boldsymbol{H} &= j\omega \boldsymbol{D} + \boldsymbol{J} \\
&= j\omega\varepsilon \boldsymbol{E} + \sigma \boldsymbol{E} \\
&= j\omega(\varepsilon' - j\varepsilon'')\boldsymbol{E} + \sigma \boldsymbol{E} \\
&= \underbrace{j\omega\varepsilon' \boldsymbol{E}}_{\text{Displacement current}} + \underbrace{(\sigma + \omega\varepsilon'')\boldsymbol{E}}_{\text{conductor +dielectric losses}}
\end{aligned}
$$

The ratio of the overall conductor and dielectric losses to the displacement current is defined as the *loss tangent* [$\tan\delta$] since it is related to the tangent of the complex number multiplying the electric field phasor.

$$\tan\delta = \frac{\sigma + \omega\varepsilon''}{\omega\varepsilon'}$$

Material Classifications

A given medium is characterized by its three constitutive parameters defined as (μ,ε,σ). We may classify media according to the characteristics of the constitutive parameters.

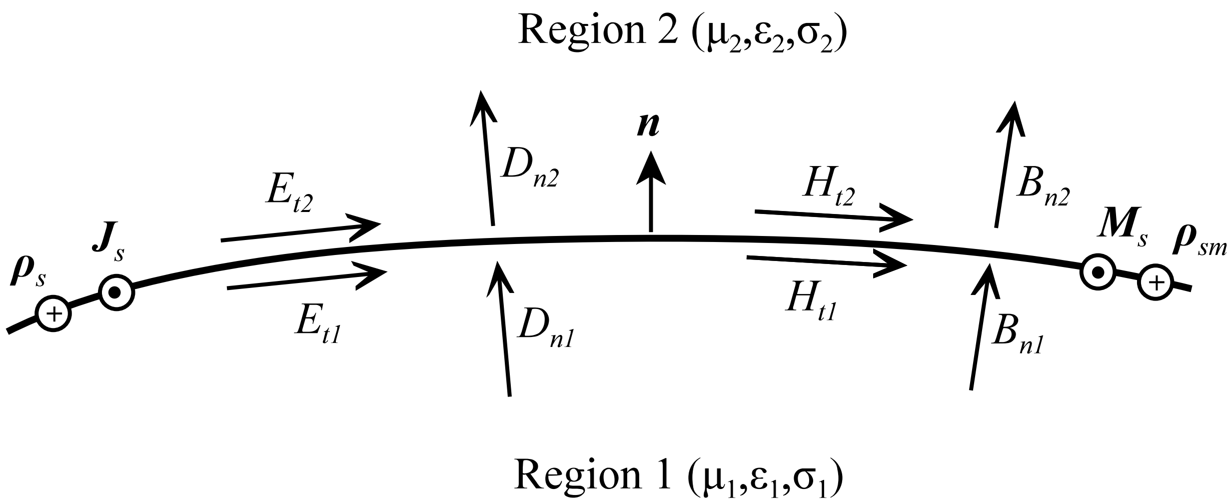
Homogeneous - the constitutive parameters of the medium are not functions of position (otherwise - *inhomogeneous*).

Linear - the constitutive parameters of the medium are not functions of the magnitude of the applied field (otherwise - *nonlinear*).

Isotropic - the constitutive parameters of the medium are not functions of the direction of the applied field (otherwise - *anisotropic*).

Electromagnetic Field Boundary Conditions

Knowledge of how the components of an electromagnetic field behave at the interface between two different media is important in the solution of many problems in microwave engineering. A simple interface between two media is shown below. The vector ***n*** is defined as the unit normal to the interface pointing into region 2.



The general boundary conditions are:

$$\boldsymbol{n} \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) = \rho_s$$

$$\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{J}_s$$

$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = \rho_{sm}$$

$$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = -\boldsymbol{M}_s$$

Note that the individual components of the vector fields [***n*** · {} defines the normal components of the vector while ***n*** × {} defines the tangential components of the vector] are discontinuous at the interface by an amount equal to the respective surface current or charge on the boundary. In most applications, we do not encounter all of the surface sources. These general boundary conditions can be specialized to problems involving specific combinations of materials.

Interface Between Two Lossless Dielectric Materials

If the two media are lossless dielectrics (perfect insulators defined by $\mu_1'' = \mu_2'' = \varepsilon_1'' = \varepsilon_2'' = \sigma_1 = \sigma_2 = 0$), then no surface charge or current will occur naturally. The boundary conditions then become

$$\boldsymbol{n} \cdot \boldsymbol{D_1} = \boldsymbol{n} \cdot \boldsymbol{D_2}$$
$$\boldsymbol{n} \times \boldsymbol{H_1} = \boldsymbol{n} \times \boldsymbol{H_2}$$
$$\boldsymbol{n} \cdot \boldsymbol{B_1} = \boldsymbol{n} \cdot \boldsymbol{B_2}$$
$$\boldsymbol{n} \times \boldsymbol{E_1} = \boldsymbol{n} \times \boldsymbol{E_2}$$

Thus, the normal components of electric and magnetic flux and the tangential components of electric and magnetic field are continuous across a lossless dielectric interface.

Perfect Electric Conductor (PEC)

If region 1 is assumed to be a perfect electric conductor ($\sigma_1 \rightarrow \infty$) while region 2 is a dielectric, no electromagnetic field can penetrate into region 1 ($\boldsymbol{E}_1 = \boldsymbol{H}_1 = 0$). Electric surface currents and charges are found on the PEC (no magnetic charge or current) which gives

$$\boldsymbol{n} \cdot \boldsymbol{D_2} = \rho_s$$
$$\boldsymbol{n} \times \boldsymbol{H_2} = \boldsymbol{J_s}$$
$$\boldsymbol{n} \cdot \boldsymbol{B_2} = 0$$
$$\boldsymbol{n} \times \boldsymbol{E_2} = \boldsymbol{0}$$

Note that the tangential electric field is always zero on the surface of a PEC. The tangential magnetic field on a PEC is equal to the surface current while the normal electric flux is equal to the surface charge.

Perfect Magnetic Conductor (PMC)

If region 1 is assumed to be a perfect magnetic conductor (its equivalent magnetic conductivity $\sigma_{m1} \rightarrow \infty$) while region 2 is a dielectric, no electromagnetic field can penetrate into region 1 ($\boldsymbol{E}_1 = \boldsymbol{H}_1 = 0$). Magnetic surface currents and charges are found on the PMC (no electric charge or current) which gives

$$\boldsymbol{n} \cdot \boldsymbol{D}_2 = 0$$

$$\boldsymbol{n} \times \boldsymbol{H}_2 = \boldsymbol{0}$$

$$\boldsymbol{n} \cdot \boldsymbol{B}_2 = \rho_{sm}$$

$$\boldsymbol{n} \times \boldsymbol{E}_2 = -\boldsymbol{M}_s$$

Note that the tangential magnetic field is always zero on the surface of a PMC. The tangential electric field on a PEC is equal to the negative of the surface magnetic current while the normal electric flux is equal to the surface magnetic charge.

Electromagnetic Waves

Maxwell's equations show that the electric field and magnetic field in a source-free ($\boldsymbol{J} = \boldsymbol{M} = \mathbf{0}$, $\rho = \rho_m = 0$), homogeneous, linear, isotropic medium satisfy wave equations (Helmholtz equations). The source-free Maxwell's equations in phasor form are

$$\nabla \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H} \qquad (1)$$

$$\nabla \times \boldsymbol{H} = j\omega\varepsilon\boldsymbol{E} \qquad (2)$$

$$\nabla \cdot \boldsymbol{E} = 0 \qquad (3)$$

$$\nabla \cdot \boldsymbol{H} = 0 \qquad (4)$$

Note that taking the divergence of (1) and (2) yields (3) and (4) since $\nabla \cdot \nabla \times \boldsymbol{F} = 0$ for any vector $\boldsymbol{F}$. Thus, in a source-free region, (3) and (4) are not necessary. Taking the curl of (1) and inserting (2) yields

$$\nabla \times \nabla \times \boldsymbol{E} = -j\omega\mu\nabla \times \boldsymbol{H} = -j\omega\mu(j\omega\varepsilon\boldsymbol{E}) = \omega^2\mu\varepsilon\boldsymbol{E} = k^2\boldsymbol{E} \qquad (5)$$

while taking the curl of (2) and inserting (1) yields

$$\nabla \times \nabla \times \boldsymbol{H} = -j\omega\mu\nabla \times \boldsymbol{E} = j\omega\varepsilon(-j\omega\mu\boldsymbol{H}) = \omega^2\mu\varepsilon\boldsymbol{H} = k^2\boldsymbol{H} \qquad (6)$$

where $k = \omega\sqrt{\mu\varepsilon}$ is defined as the *wavenumber* of the medium. Using the vector identity

$$\nabla \times \nabla \times \boldsymbol{F} = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2\boldsymbol{F} \qquad \text{for any vector } \mathbf{F}$$

in (5) and (6) gives

$$\nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2\boldsymbol{E} - k^2\boldsymbol{E} = \mathbf{0} \qquad (7)$$

$$\nabla(\nabla \cdot \boldsymbol{H}) - \nabla^2\boldsymbol{H} - k^2\boldsymbol{H} = \mathbf{0} \qquad (8)$$

However, the divergence terms in (7) and (8) are zero in the source-free region. This gives the wave equations (Helmholtz equations) for the electric and magnetic field.

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = \mathbf{0}$$

$$\nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = \mathbf{0}$$

Wave equations (Helmholtz Equations)

The wave equations for the $\boldsymbol{E}$ and $\boldsymbol{H}$ show that energy will propagate away from a time-varying electromagnetic source in the form of electromagnetic waves.

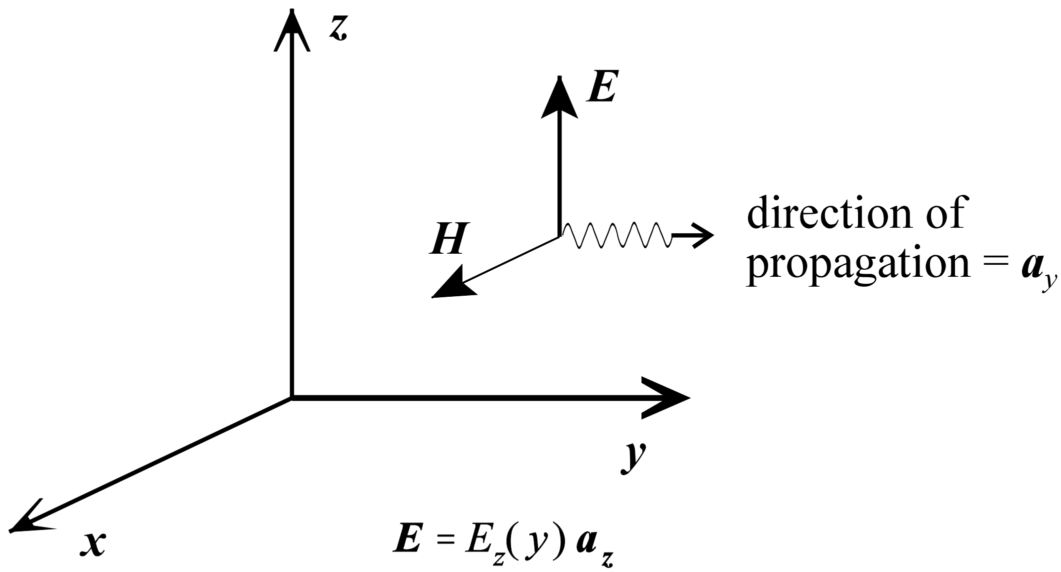
Plane Waves

Plane waves are the most commonly encountered wave type in electromagnetic applications and are the easiest to define mathematically.

Plane wave - the electric and magnetic field of a plane wave lie in the plane which is perpendicular to the direction of wave propagation (the direction of $\boldsymbol{E} \times \boldsymbol{H}$ is the direction of wave propagation).

Uniform Plane wave - the electric and magnetic fields of a uniform plane wave are uniform in the plane which is perpendicular to the direction of propagation (the magnitude of $\boldsymbol{E}$ and $\boldsymbol{H}$ vary only in the direction of wave propagation).

Example (Uniform plane wave)



$$\boldsymbol{E} = E_z(y)\,\boldsymbol{a}_z$$

$$\boldsymbol{H} = H_x(y)\,\boldsymbol{a}_x$$

The uniform plane wave for this example has only a z-component of electric field and an x-component of magnetic field which are both functions of only y. The vector Laplacian operator (∇^2) which appears in the wave equations for $\boldsymbol{E}$ and $\boldsymbol{H}$ may be expanded in rectangular coordinates as

$$\begin{aligned}
\nabla^2\boldsymbol{F} &= \nabla^2 F_x\,\boldsymbol{a}_x + \nabla^2 F_y\,\boldsymbol{a}_y + \nabla^2 F_z\,\boldsymbol{a}_z \\
&= \left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_x}{\partial z^2}\right)\boldsymbol{a}_x \\
&+ \left(\frac{\partial^2 F_y}{\partial x^2} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_y}{\partial z^2}\right)\boldsymbol{a}_y \\
&+ \left(\frac{\partial^2 F_z}{\partial x^2} + \frac{\partial^2 F_z}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2}\right)\boldsymbol{a}_z
\end{aligned}$$

Given the vector Laplacian definition, the wave equations for $\boldsymbol{E}$ and $\boldsymbol{H}$ reduce to

$$\nabla^2\boldsymbol{E} + k^2\boldsymbol{E} = \frac{d^2E_z}{dy^2}\boldsymbol{a_z} + k^2E_z\boldsymbol{a_z} = \mathbf{0}$$

$$\nabla^2\boldsymbol{H} + k^2\boldsymbol{H} = \frac{d^2H_x}{dy^2}\boldsymbol{a_x} + k^2H_x\boldsymbol{a_x} = \mathbf{0}$$

where the partial derivatives have been replaced by pure derivatives given that the field components are functions of only one variable. Note that the right hand side of the equations above is the zero vector. Thus, by equating the vector components on both sides of the equation, we may write scalar equations for E_z and H_x.

$$\frac{d^2E_z}{dy^2} + k^2E_z = 0$$

$$\frac{d^2H_x}{dy^2} + k^2H_x = 0$$

Linear, homogeneous, 2nd order D.E.'s

The general solutions to these D.E.'s are

$$E_z(y) = E_1e^{jky} + E_2e^{-jky}$$

$$H_x(y) = H_1e^{jky} + H_2e^{-jky}$$

where E_1, E_2, H_1, and H_2 are constants. The instantaneous forms of the wave field components are

$$\mathscr{E}_z\,(y,t) = Re\left\{\left[E_1e^{jky} + E_2e^{-jky}\right]e^{j\omega t}\right\}$$
$$= E_1\cos(\omega t + ky) + E_2\cos(\omega t - ky)$$

$$\mathscr{H}_x\ (y,t) = Re\left\{\left[H_1 e^{jky} + H_2 e^{-jky}\right] e^{j\omega t}\right\}$$

$$= H_1 \cos(\omega t + ky) + H_2 \cos(\omega t - ky)$$

The direction of propagation for the plane wave may be determined by investigating the points of constant phase on the waves.

$$\omega t + ky = constant \qquad -\boldsymbol{a}_y \text{ traveling wave}$$
$$(\text{as } t\uparrow,\ y\downarrow) \qquad (e^{jky})$$

$$\omega t - ky = constant \qquad +\boldsymbol{a}_y \text{ traveling wave}$$
$$(\text{as } t\uparrow,\ y\uparrow) \qquad (e^{-jky})$$

Given the $+\boldsymbol{a}_y$ traveling wave of our example, the constants E_1 and H_1 must be zero so that

$$E_z(y) = E_2 e^{-jky} \qquad \text{or} \qquad \boldsymbol{E}(y) = E_2 e^{-jky}\,\boldsymbol{a}_z$$

$$H_x(y) = H_2 e^{-jky} \qquad \text{or} \qquad \boldsymbol{H}(y) = H_2 e^{-jky}\,\boldsymbol{a}_x$$

Plane wave parameters

The velocity of propagation (v_p) of the plane wave is found by differentiating the position of the point of constant phase with respect to position.

$$\omega t - ky = constant \qquad \Rightarrow \qquad y = \frac{1}{k}(\omega t - constant)$$

$$v_p = \frac{dy}{dt} = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}} \qquad (\text{m/s})$$

In free space, $v_p = 1/\sqrt{\mu_o\varepsilon_o} = c$ (speed of light = 3×10^8 m/s).

In media with $\mu_r > 1$ and/or $\varepsilon_r > 1$, $v_p < c$.

The radian frequency of the plane wave is defined by

$$\omega = 2\pi f = 2\pi \frac{1}{T} \qquad \text{(rad/s)}$$

With the wave traveling at a velocity of v_p, it takes one period (T) for the wave to travel one wavelength (λ).

$$\lambda = v_p T = \frac{v_p}{f} = \frac{\omega / k}{f} = \frac{2\pi}{k} \qquad \text{(m)}$$

$$k = \frac{2\pi}{\lambda} \qquad (\text{m}^{-1})$$

The wavenumber definition in terms of λ shows that the all waves see a phase change of 2π radians per wavelength.

Plane waves have the characteristic that the ratio of the electric field to magnetic field at any point is a constant which is related to the constitutive parameters of the medium. This property can be illustrated by using Maxwell's equations with our example plane wave.

$$\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H}$$

$$\boldsymbol{H} = \frac{-1}{j\omega\mu} \nabla \times \boldsymbol{E} = \frac{-1}{j\omega\mu} \left[\frac{\partial E_z}{\partial y} \boldsymbol{a_x} - \frac{\partial E_z}{\partial x} \boldsymbol{a_y} \right] = \frac{-1}{j\omega\mu} \frac{\partial}{\partial y} \left(E_2 e^{-jky} \right) \boldsymbol{a_x}$$

$$= \frac{-1}{j\omega\mu} (-jkE_2 e^{-jky})\, \boldsymbol{a_x} = \frac{k}{\omega\mu} E_2 e^{-jky} \boldsymbol{a_x} = H_2 e^{-jky} \boldsymbol{a_x}$$

$$\frac{E_2}{H_2} = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} = \eta \qquad \text{[wave impedance } (\Omega)]$$

In free space, the wave impedance is $\sqrt{\mu_o/\varepsilon_o} \approx 120\,\pi = 377\ \Omega$.

Plane Waves in Lossy Media

A plane wave loses energy as it propagates through a lossy medium. A medium is defined as a *lossy medium* if it is characterized by any or all of the following loss mechanisms:

conduction losses $\Rightarrow$ $(\sigma > 0)$

dielectric losses $\Rightarrow$ $(\varepsilon'' > 0)$

magnetic losses $\Rightarrow$ $(\mu'' > 0)$

Dielectric and magnetic losses are typically small and can be neglected for most materials. However, conduction losses can be significant for commonly encountered materials.

If we include conduction losses in a homogeneous, isotropic, linear medium while assuming that the dielectric and magnetic losses are negligible (μ and ϵ are real), Maxwell's equations become

$$\nabla \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H} \tag{1}$$

$$\nabla \times \boldsymbol{H} = j\omega\varepsilon\boldsymbol{E} + \underbrace{\sigma\boldsymbol{E}}_{\text{conduction losses}} \tag{2}$$

$$\nabla \cdot \boldsymbol{E} = 0 \tag{3}$$

$$\nabla \cdot \boldsymbol{H} = 0 \tag{4}$$

Following the same techniques used in the lossless problem, we find that $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy wave equations (5) and (6) which include a complex *propagation constant* γ (as opposed to a real wavenumber in the lossless case).

$$\nabla^2\boldsymbol{E} - (j\omega\mu)(\sigma + j\omega\varepsilon)\boldsymbol{E} = \nabla^2\boldsymbol{E} - \gamma^2\boldsymbol{E} = \mathbf{0} \tag{5}$$

$$\nabla^2\boldsymbol{H} - (j\omega\mu)(\sigma + j\omega\varepsilon)\boldsymbol{H} = \nabla^2\boldsymbol{H} - \gamma^2\boldsymbol{H} = \mathbf{0} \tag{6}$$

$$\gamma^2 = (j\omega\mu)(\sigma + j\omega\varepsilon) = -\omega^2\mu\varepsilon + j\omega\mu\sigma = -\omega^2\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)$$

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}} = jk\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}} \qquad (m^{-1})$$

γ - propagation constant

α - attenuation constant

β - phase constant

Note that the propagation constant reduces to $\gamma = jk$ ($\beta = k$) when $\sigma = 0$. The solutions for the attenuation and phase constants in terms of μ, ε and σ are

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$$

Given the same +***y***-directed uniform plane wave assumed in the lossless example, the differential equations governing the plane wave field components in the lossy medium are

$$\frac{d^2E_z}{dy^2} - \gamma^2 E_z = 0 \qquad\qquad \frac{d^2H_x}{dy^2} - \gamma^2 H_x = 0$$

which have general solutions of the form

$$E_z(y) = E_1 e^{\gamma y} + E_2 e^{-\gamma y} = E_1 e^{\alpha y} e^{j\beta y} + E_2 e^{-\alpha y} e^{-j\beta y}$$

$$H_x(y) = H_1 e^{\gamma y} + H_2 e^{-\gamma y} = \underbrace{H_1 e^{\alpha y} e^{j\beta y}}_{-\boldsymbol{y}\text{ directed wave}} + \underbrace{H_2 e^{-\alpha y} e^{-j\beta y}}_{+\boldsymbol{y}\text{ directed wave}}$$

The instantaneous fields of the plane wave in a lossy medium are

$$\mathscr{E}(y,t) = Re\left\{\boldsymbol{E}(y)e^{j\omega t}\right\} = E_1 e^{\alpha y}\cos(\omega t + \beta y) + E_2 e^{-\alpha y}\cos(\omega t - \beta y)$$

$$\mathscr{H}(y,t) = Re\left\{\boldsymbol{H}(y)e^{j\omega t}\right\} = H_1 e^{\alpha y}\cos(\omega t + \beta y) + H_2 e^{-\alpha y}\cos(\omega t - \beta y)$$

Since the particular solution contains only a $+\boldsymbol{y}$ traveling wave, the constants E_1 and H_1 must be zero.

$$E_z(y) = E_2 e^{-\gamma y} \qquad \text{or} \qquad \boldsymbol{E}(y) = E_2 e^{-\gamma y}\boldsymbol{a}_z$$

$$H_x(y) = H_2 e^{-\gamma y} \qquad \text{or} \qquad \boldsymbol{H}(y) = H_2 e^{-\gamma y}\boldsymbol{a}_x$$

Note that the phase constant β defines the phase associated with the plane wave propagating in a lossy medium. The resulting equations for the wave parameters must be adjusted accordingly (replace k with β).

$$v_p = \frac{\omega}{\beta} \qquad\qquad \lambda = \frac{2\pi}{\beta}$$

The wave impedance in the lossy medium is complex as shown using Maxwell's equations.

$$\nabla \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H}$$

$$\boldsymbol{H} = \frac{-1}{j\omega\mu}\nabla\times\boldsymbol{E} = \frac{-1}{j\omega\mu}\left(\frac{\partial E_z}{\partial y}\boldsymbol{a}_x\right) = \frac{-1}{j\omega\mu}\frac{\partial}{\partial y}\left(E_2 e^{-\gamma y}\right)\boldsymbol{a}_x$$

$$= \frac{-1}{j\omega\mu}(-\gamma E_2 e^{-\gamma y})\boldsymbol{a}_x = \frac{\gamma}{j\omega\mu}E_2 e^{-\gamma y}\boldsymbol{a}_x = H_2 e^{-\gamma y}\boldsymbol{a}_x$$

$$\frac{E_2}{H_2} = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{(j\omega\mu)(\sigma + j\omega\varepsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \eta$$

Alternatively, the conduction losses may be included in the complex permittivity as defined by the loss tangent.

$$
\begin{aligned}
\nabla \times \boldsymbol{H} &= j\omega\varepsilon \boldsymbol{E} + \sigma \boldsymbol{E} \\
&= j\omega(\varepsilon' - j\varepsilon'')\boldsymbol{E} + \sigma \boldsymbol{E} \\
&= j\omega\left[\varepsilon' - j\varepsilon'' + \frac{\sigma}{j\omega}\right]\boldsymbol{E} \\
&= j\omega\left[\varepsilon'\left(1 - j\frac{\varepsilon''}{\varepsilon'} - j\frac{\sigma}{\omega\varepsilon'}\right)\right]\boldsymbol{E} \\
&= j\omega\left\{\varepsilon'\left[1 - j\left(\frac{\omega\varepsilon'' + \sigma}{\omega\varepsilon'}\right)\right]\right\}\boldsymbol{E} \\
&= j\omega\left[\varepsilon'(1 - j\tan\delta)\right]\boldsymbol{E}
\end{aligned}
$$

Thus, using the effective complex permittivity term in brackets above, the conductor and dielectric losses may be included without explicitly writing a conduction current term.

Plane Waves in Good Conductors

For good conductors ($\sigma >> \omega\varepsilon$), the propagation constant may be approximated by

$$\gamma = \alpha + j\beta = \sqrt{(j\omega\mu)(\sigma + j\omega\varepsilon)} \approx \sqrt{j\omega\mu\sigma} = (1 + j)\sqrt{\frac{\omega\mu\sigma}{2}}$$

The inverse of the attenuation constant for good conductors is defined as the *skin depth* δ_s. The skin depth defines the distance over which a plane traveling in a good conductor wave decays by an amount of $e^{-1} = 0.368$.

$$\alpha = \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} = \frac{1}{\delta_s}$$

The wave impedance within a good conductor is

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \approx \sqrt{\frac{j\omega\mu}{\sigma}} = (1 + j)\sqrt{\frac{\omega\mu}{2\sigma}} = \frac{(1 + j)}{\sigma\delta_s}$$

Poynting's Theorem

Poynting's theorem is the basic conservation law for electromagnetic energy. It defines the balance of complex power given sources of electromagnetic energy, energy storage and dissipation. The direction and density of electromagnetic power flow at a point is defined by the *Poynting vector*. The instantaneous form of the Poynting vector $\mathscr{S}$ is

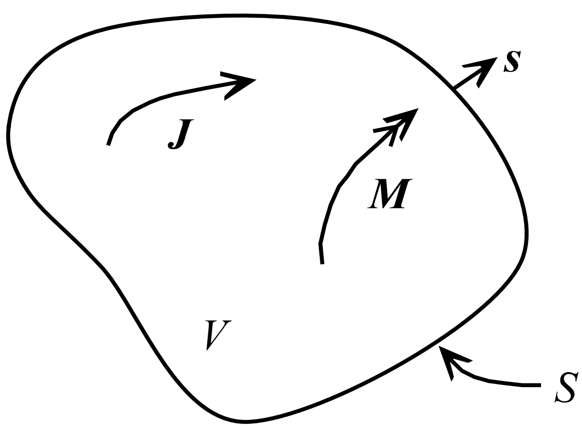
$$\mathscr{S} = \mathscr{E} \times \mathscr{H}$$

The corresponding phasor form of the Poynting vector $\boldsymbol{S}$ is

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}^*$$

Given a volume V enclosed by the surface S which contains electric and magnetic sources $\boldsymbol{J}$ and $\boldsymbol{M}$, Poynting's theorem for the volume may be written as

$$P_s = P_o + P_l + j2\omega(W_m - W_e) \qquad \text{(Poynting's theorem)}$$



$$P_s = -\frac{1}{2}\int_V (\boldsymbol{E}\cdot\boldsymbol{J}^* + \boldsymbol{H}^*\cdot\boldsymbol{M})\,dv$$ Complex power delivered by the sources

$$P_o = \frac{1}{2}\int_S (\boldsymbol{E}\times\boldsymbol{H}^*)\cdot d\boldsymbol{s}$$ Complex power flow out of the volume V

$$P_l = \frac{\sigma}{2}\int_V |\boldsymbol{E}|^2\, dv + \frac{\omega}{2}\int_V (\varepsilon''|\boldsymbol{E}|^2 + \mu''|\boldsymbol{H}|^2)\, dv$$

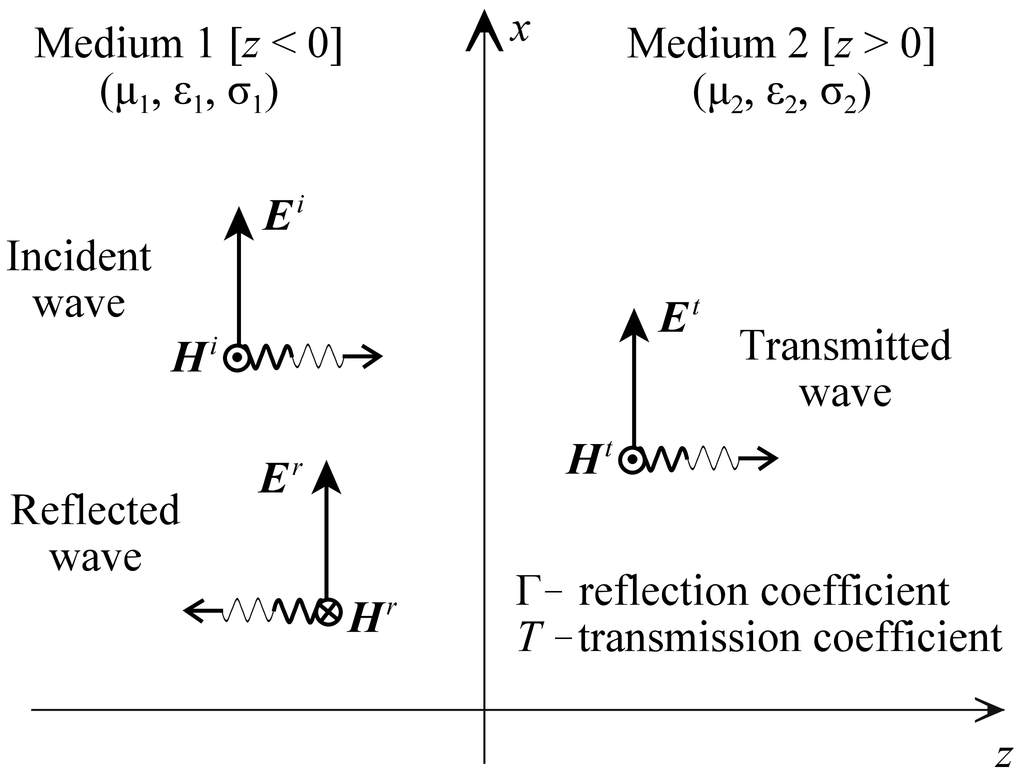
$$\underbrace{\qquad\qquad}_{\text{Conduction losses}} \qquad \underbrace{\qquad\qquad}_{\text{Dielectric losses}} \qquad \underbrace{\qquad\qquad}_{\text{Magnetic losses}}$$

$$\underbrace{W_e = \frac{1}{4}\int_V \varepsilon' |\boldsymbol{E}|^2\, dv}_{\text{Stored electric energy}} \qquad \underbrace{W_m = \frac{1}{4}\int_V \mu' |\boldsymbol{H}|^2\, dv}_{\text{Stored magnetic energy}}$$

Poynting's theorem states that the complex power produced by the sources is equal to the power transmitted out of the volume plus that dissipated in the form of heat (through conductor, dielectric and magnetic losses) plus the 2ω times the net reactive stored energy.

Plane Wave Reflection/Transmission at a Planar Interface

Normal Incidence



Incident wave fields

$$\boldsymbol{E^i} = E_o e^{-\gamma_1 z} \boldsymbol{a_x}$$

$$\boldsymbol{H^i} = \frac{E_o}{\eta_1} e^{-\gamma_1 z} \boldsymbol{a_y}$$

Reflected wave fields

$$\boldsymbol{E^r} = \Gamma E_o e^{\gamma_1 z} \boldsymbol{a_x}$$

$$\boldsymbol{H^r} = -\Gamma \frac{E_o}{\eta_1} e^{\gamma_1 z} \boldsymbol{a_y}$$

Transmitted wave fields

$$\boldsymbol{E^t} = T E_o e^{-\gamma_2 z} \boldsymbol{a_x}$$

$$\boldsymbol{H^t} = T \frac{E_o}{\eta_2} e^{-\gamma_2 z} \boldsymbol{a_y}$$

Boundary Conditions

$$E_x^i + E_x^r = E_x^t \qquad at \quad z = 0 \qquad \Rightarrow \qquad 1 + \Gamma = T$$

$$H_x^i + H_x^r = H_x^t \qquad at \quad z = 0 \qquad \Rightarrow \qquad \frac{1 - \Gamma}{\eta_1} = \frac{T}{\eta_2}$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \text{(Reflection coefficient)}$$

$$T = \frac{2\eta_2}{\eta_2 + \eta_1} \qquad \text{(Transmission coefficient)}$$

Surface Impedance of a Good Conductor

If region 1 is air and region 2 is a good conductor, the wave is attenuated rapidly as it penetrates the conductor. The electric field within the good conductor is given by

$$\boldsymbol{E_2} = T E_o e^{-\gamma_2 z} \boldsymbol{a_x} \qquad \text{(V/m)}$$

where the propagation constant in the conductor may be written in terms of the skin depth δ_s as

$$\gamma_2 \approx \frac{(1 + j)}{\delta_s} \qquad (\text{m}^{-1})$$

The conduction current density within the conductor is given by

$$\boldsymbol{J_2} = \sigma_2 \boldsymbol{E_2} = \sigma_2 T E_o e^{-\gamma_2 z} \boldsymbol{a_x} \qquad (\text{A/m}^2)$$

We may determine the total current per unit width (y-direction) by integrating the current density over all z. Given this value, we may assume that this total current is spread uniformly from the surface of the conductor to a depth δ_s.

$$\boldsymbol{J_{2s}} = \int_0^\infty \boldsymbol{J_2}(z)\,dz = \sigma_2 T E_o \boldsymbol{a_x} \int_0^\infty e^{-\gamma_2 z}\,dz = \frac{\sigma_2 T E_o}{\gamma_2}\boldsymbol{a_x} \qquad \text{(A/m)}$$

The transmission coefficient for the air-good conductor example is

$$T = \frac{2\eta_2}{\eta_2 + \eta_1} \qquad \eta_1 = \eta_o \qquad \eta_2 \approx \frac{(1+j)}{\sigma_2 \delta_s}$$

Since $|\eta_2| << |\eta_1|$, the transmission coefficient may be written as

$$T \approx \frac{2\eta_2}{\eta_1} = \frac{2(1+j)}{\eta_o \sigma_2 \delta_s}$$

Using the equations for γ_2 and T in the equation for $\boldsymbol{J}_{2s}$ gives

$$\boldsymbol{J_{2s}} = \sigma_2 E_o \frac{\delta_s}{1+j}\,\frac{2(1+j)}{\eta_o \sigma \delta_s} = \frac{2E_o}{\eta_o}$$

Our approximation for the current density within conductor becomes

$$\boldsymbol{J_2} \approx \begin{cases} \dfrac{\boldsymbol{J_{2s}}}{\delta_s} & 0 \le z \le \delta_s \\ 0 & \delta_s \le z \le \infty \end{cases}$$

The power dissipated within the conductor is given by

$$P_2 = \frac{1}{2}\int_V \boldsymbol{E_2}\cdot\boldsymbol{J_2^*}\,dv = \frac{1}{2}\int_V \frac{\boldsymbol{J_2}}{\sigma_2}\cdot\boldsymbol{J_2^*}\,dv = \frac{1}{2\sigma_2}\int_V |\boldsymbol{J_2}|^2\,dv$$

$$= \frac{1}{2\sigma_2}\int_S\int_0^{\delta_s} \frac{|\boldsymbol{J_{2s}}|^2}{\delta_s^2}\,dv = \frac{4|E_o|^2}{2\sigma_2\eta_o\delta_s^2}(S\delta_s) = \frac{2|E_o|^2 S}{\eta_o^2\sigma_2\delta_s}$$

We may express the dissipated power as

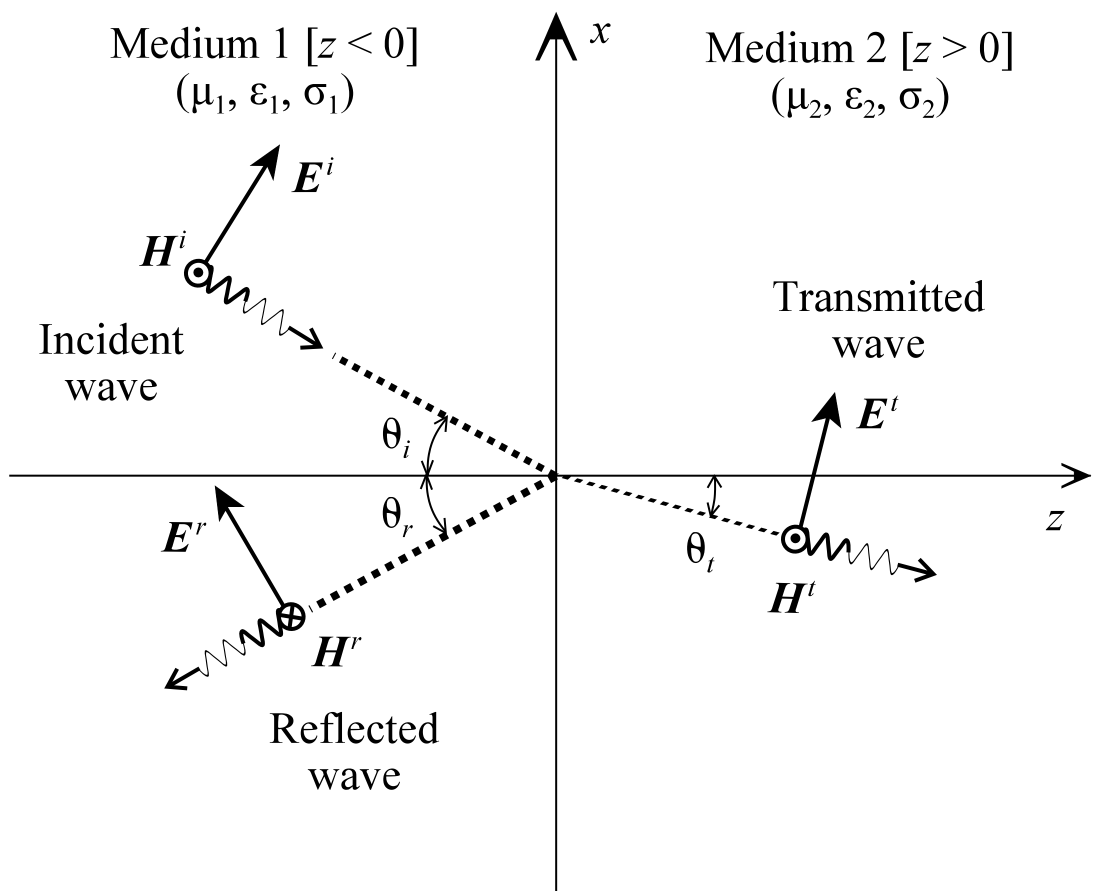
$$P_2 = \frac{2\,|E_o|^2 S}{\eta_o^2 \sigma_2 \delta_s} = \frac{2\,|E_o|^2}{\eta_o^2} S R_s$$

where R_s is defined as the surface resistance of the conductor and is given by

$$R_s = \frac{1}{\sigma_2 \delta_s}$$

<u>Oblique Incidence</u> (assume lossless dielectrics)

<u>Parallel Polarization</u>



θ_i - angle of incidence

θ_r - angle of reflection

θ_t - angle of transmission

Application of the boundary conditions at the interface yields

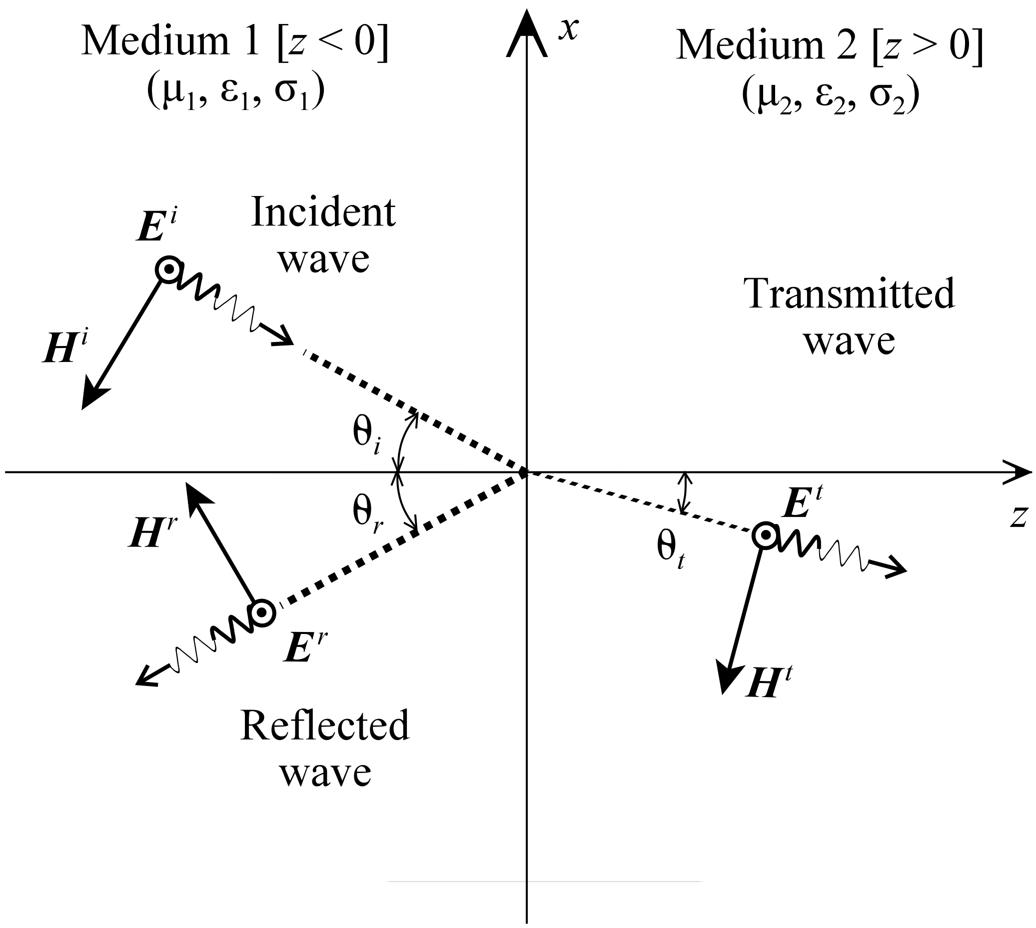
$$\theta_r = \theta_i$$

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_1}{k_2}$$

$$\Gamma_{\parallel} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$T_{\parallel} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

Perpendicular Polarization

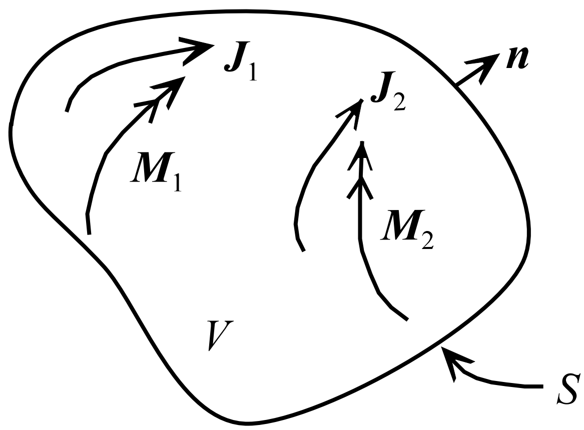


$$\Gamma_{\perp} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$T_{\perp} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

Reciprocity Theorem

The reciprocity theorem is a useful mathematical tool used to recast certain electromagnetic problems into different forms. Assume that a volume V enclosed by the surface S contains two sets of sources ($\boldsymbol{J}_1$, $\boldsymbol{M}_1$) and ($\boldsymbol{J}_2$, $\boldsymbol{M}_2$). The reciprocity theorem relates the response at one source due to the second source to the response at the second source due to the first source.



Starting with Maxwell's equations defining the field responses to the two sets of sources

$$\nabla \times \boldsymbol{E}_1 = -j\omega\mu \boldsymbol{H}_1 - \boldsymbol{M}_1$$

$$\nabla \times \boldsymbol{H}_1 = j\omega\varepsilon \boldsymbol{E}_1 + \boldsymbol{J}_1$$

$$\nabla \times \boldsymbol{E}_2 = -j\omega\mu \boldsymbol{H}_2 - \boldsymbol{M}_2$$

$$\nabla \times \boldsymbol{H}_2 = j\omega\varepsilon \boldsymbol{E}_2 + \boldsymbol{J}_2$$

we may use vector identities to relate these responses.

$$\nabla \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2) = (\nabla \times \boldsymbol{E}_1) \cdot \boldsymbol{H}_2 - (\nabla \times \boldsymbol{H}_2) \cdot \boldsymbol{E}_1$$

$$= -j\omega\mu \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 - \boldsymbol{M}_1 \cdot \boldsymbol{H}_2 - j\omega\varepsilon \boldsymbol{E}_2 \cdot \boldsymbol{E}_1 - \boldsymbol{J}_2 \cdot \boldsymbol{E}_1$$

$$\nabla \cdot (\boldsymbol{E_2} \times \boldsymbol{H_1}) = (\nabla \times \boldsymbol{E_2}) \cdot \boldsymbol{H_1} - (\nabla \times \boldsymbol{H_1}) \cdot \boldsymbol{E_2}$$

$$= -j\omega\mu \boldsymbol{H_2} \cdot \boldsymbol{H_1} - \boldsymbol{M_2} \cdot \boldsymbol{H_1} - j\omega\varepsilon \boldsymbol{E_1} \cdot \boldsymbol{E_2} - \boldsymbol{J_1} \cdot \boldsymbol{E_2}$$

If we subtract the two divergence equations, we find

$$\nabla \cdot (\boldsymbol{E_1} \times \boldsymbol{H_2}) - \nabla \cdot (\boldsymbol{E_2} \times \boldsymbol{H_1}) = (\boldsymbol{E_2} \cdot \boldsymbol{J_1} - \boldsymbol{H_2} \cdot \boldsymbol{M_1}) - (\boldsymbol{E_1} \cdot \boldsymbol{J_2} - \boldsymbol{H_1} \cdot \boldsymbol{M_2})$$

We may integrate both sides of the equation above over the volume V and apply the divergence theorem to find

$$\int_V \nabla \cdot (\boldsymbol{E_1} \times \boldsymbol{H_2} - \boldsymbol{E_2} \times \boldsymbol{H_1})\, dv = \int_S (\boldsymbol{E_1} \times \boldsymbol{H_2} - \boldsymbol{E_2} \times \boldsymbol{H_1}) \cdot d\boldsymbol{s}$$

or

$$\int_V \nabla \cdot (\boldsymbol{E_1} \times \boldsymbol{H_2} - \boldsymbol{E_2} \times \boldsymbol{H_1})\, dv = \int_S (\boldsymbol{E_1} \times \boldsymbol{H_2} - \boldsymbol{E_2} \times \boldsymbol{H_1}) \cdot d\boldsymbol{s}$$

$$= \int_V [(\boldsymbol{E_2} \cdot \boldsymbol{J_1} - \boldsymbol{H_2} \cdot \boldsymbol{M_1}) - (\boldsymbol{E_1} \cdot \boldsymbol{J_2} - \boldsymbol{H_1} \cdot \boldsymbol{M_2})]\, dv$$

Source-Free Region

If S is a source free region ($\boldsymbol{J}_1 = \boldsymbol{M}_1 = \boldsymbol{J}_2 = \boldsymbol{M}_2 = 0$), the integral reduces to

$$\int_S (\boldsymbol{E_1} \times \boldsymbol{H_2}) \cdot d\boldsymbol{s} = \int_S (\boldsymbol{E_2} \times \boldsymbol{H_1}) \cdot d\boldsymbol{s}$$

Enclosed PEC Structure

If S is a an enclosed PEC structure, then

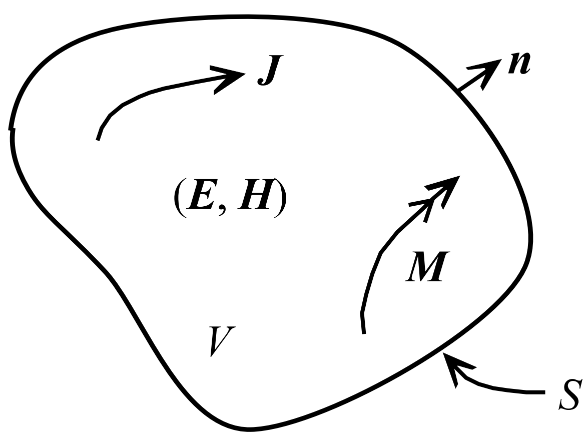
$$\boldsymbol{n} \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2) = (\boldsymbol{n} \times \boldsymbol{E}_1) \cdot \boldsymbol{H}_2 = 0$$

$$\boldsymbol{n} \cdot (\boldsymbol{E}_2 \times \boldsymbol{H}_1) = (\boldsymbol{n} \times \boldsymbol{E}_2) \cdot \boldsymbol{H}_1 = 0$$

$$\int_V (\boldsymbol{E}_2 \cdot \boldsymbol{J}_1 - \boldsymbol{H}_2 \cdot \boldsymbol{M}_1)\, dv = \int_V (\boldsymbol{E}_1 \cdot \boldsymbol{J}_2 - \boldsymbol{H}_1 \cdot \boldsymbol{M}_2)\, dv$$

Uniqueness Theorem

Given a volume V enclosed by a surface S which is completely filled with lossy media, the fields withing S are uniquely determined by the sources within S and the tangential components of $\boldsymbol{E}$ or $\boldsymbol{H}$ on S.

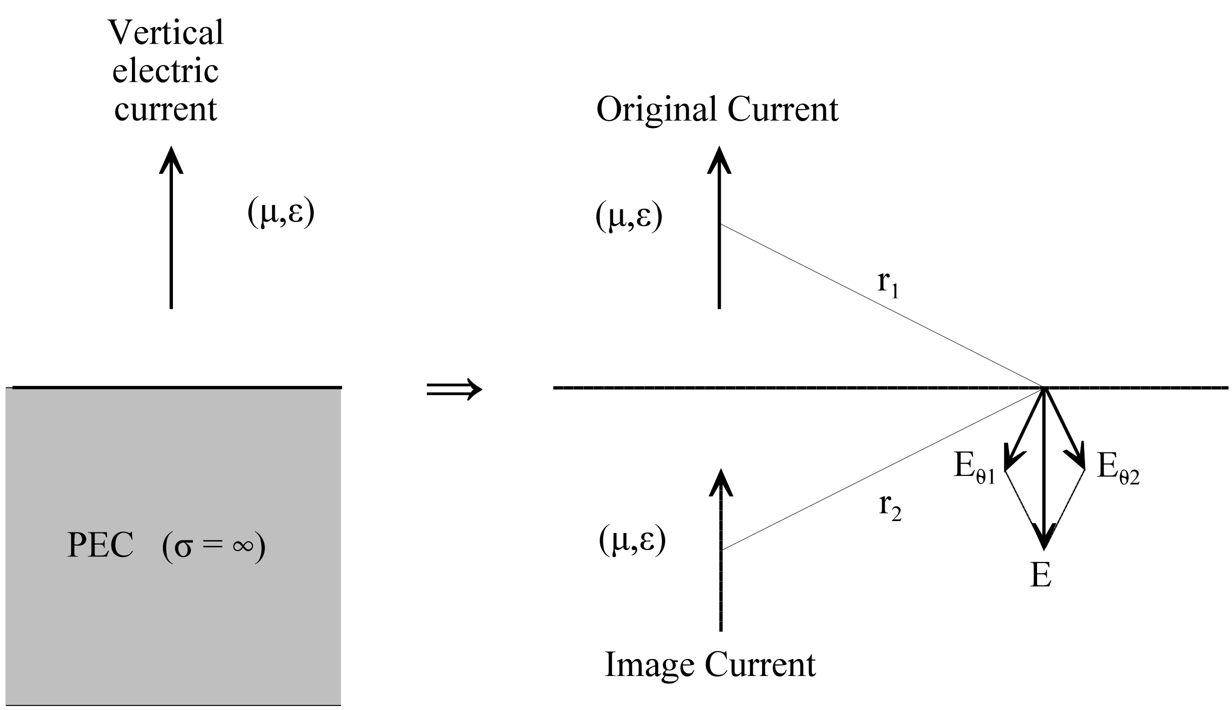


When solving boundary value problems, if we find a solution to Maxwell's equations which satisfies the appropriate boundary conditions, the uniqueness theorem ensures that the solution is unique.

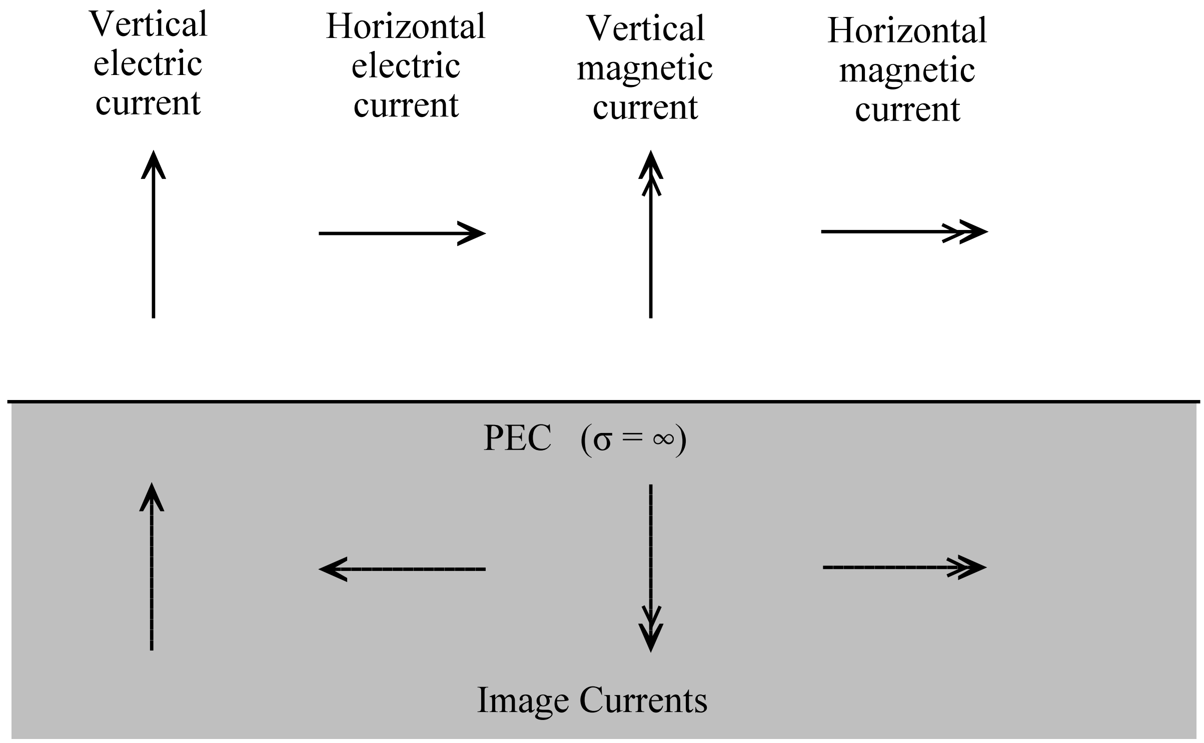
Image Theory

Given a current in the presence of a perfect conducting ground plane, [perfect electric conductor (PEC), perfect magnetic conductor (PMC)] we may use image theory to formulate the total fields without ever having to determine the surface currents induced on the ground plane. Image theory is based on the electric or magnetic field boundary condition on the surface of the perfect conductor (the tangential electric field is zero on the surface of a PEC, the tangential magnetic field is zero on the surface of a PMC). Using image theory, the ground plane can be replaced by the equivalent image current located an equal distance below the ground plane. The original current and its image are now radiating in a homogeneous medium of infinite extent and we may use the corresponding homogeneous medium equations.

Example (vertical electric current)



Currents over a PEC



Currents over a PMC