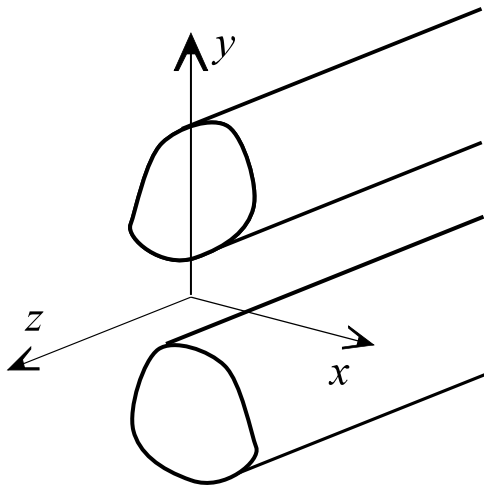
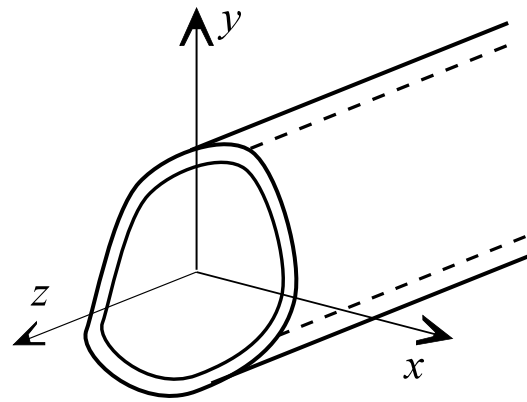


Transmission Lines and Waveguides

Given a particular conductor geometry for a transmission line or waveguide, only certain patterns of electric and magnetic fields (*modes*) can exist for propagating waves. These modes must be solutions to the governing differential equation (wave equation) while satisfying the appropriate boundary conditions for the fields.



Transmission line



Waveguide

- | | |
|--|--|
| <ul style="list-style-type: none">• Two or more conductors (two-wire, coaxial, etc.).• Can define a unique current and voltage and characteristic impedance along the line (use circuit equations). | <ul style="list-style-type: none">• Typically one enclosed conductor (rectangular, circular, etc.).• Cannot define a unique voltage and current along the waveguide (must use field equations.) |
|--|--|

The propagating modes along the transmission line or waveguide may be classified according to which field components are present or not present in the wave. The field components in the direction of wave propagation are defined as *longitudinal* components while those perpendicular to the direction of propagation are defined as *transverse* components.

Assuming the transmission line or waveguide is oriented with its axis along the z -axis (direction of wave propagation), the modes may be classified as

- (1) *Transverse electromagnetic (TEM) modes* - the electric and magnetic fields are transverse to the direction of wave propagation with no longitudinal components [$E_z = H_z = 0$]. TEM modes cannot exist on single conductor guiding structures. TEM modes are sometimes called *transmission line modes* since they are the dominant modes on transmission lines. Plane waves can also be classified as TEM modes.

Quasi-TEM modes - modes which approximate true TEM modes when the frequency is sufficiently small.

$$\lim_{f \rightarrow 0} E_z = \lim_{f \rightarrow 0} H_z = 0$$

- (2) *Transverse electric (TE) modes* - the electric field is transverse to the direction of propagation (no longitudinal electric field component) while the magnetic field has both transverse and longitudinal components [$E_z = 0, H_z \neq 0$].
- (3) *Transverse magnetic (TM) modes* - the magnetic field is transverse to the direction of propagation (no longitudinal magnetic field component) while the electric field has both transverse and longitudinal components [$H_z = 0, E_z \neq 0$].

TE and TM modes are commonly referred to as *waveguide modes* since they are the only modes which can exist in an enclosed guiding structure. TE and TM modes are characterized by a *cutoff frequency* below which they do not propagate. TE and TM modes can exist on transmission lines but are generally undesirable. Transmission lines are typically operated at frequencies below the cutoff frequencies of TE and TM modes so that only the TEM mode exists.

- (4) *Hybrid modes* (EH or HE modes) - both the electric and magnetic fields have longitudinal components [$H_z \neq 0, E_z \neq 0$]. The longitudinal electric field is dominant in the EH mode

while the longitudinal magnetic field is dominant in the HE mode. Hybrid modes are commonly found in waveguides with inhomogeneous dielectrics and optical fibers.

General Guided Wave Solutions

We may write general solutions to the fields associated with the waves that propagate on a guiding structure using Maxwell's equations. We assume the following about the guiding structure:

- (1) the guiding structure is infinitely long, oriented along the z -axis, and uniform along its length.
- (2) the guiding structure is constructed from ideal materials (conductors are PEC and insulators are lossless).
- (3) fields are time-harmonic.

The fields of the guiding structure must satisfy the source free Maxwell's equations given by

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$$

For a wave propagating along the guiding structure in the z -direction, the associated electric and magnetic fields may be written as

$$\mathbf{E}(x,y,z) = [\mathbf{e}(x,y) + e_z(x,y)\mathbf{a}_z] e^{-j\beta z}$$

$$\mathbf{H}(x,y,z) = [\mathbf{h}(x,y) + h_z(x,y)\mathbf{a}_z] e^{-j\beta z}$$

The vectors $\mathbf{e}(x,y)$ and $\mathbf{h}(x,y)$ represent the transverse field components of the wave while vectors $e_z(x,y)\mathbf{a}_z$ and $h_z(x,y)\mathbf{a}_z$ are the longitudinal components of the wave. By expanding the curl operator in rectangular coordinates, and noting that the derivatives of the transverse components with respect to z can be evaluated as

$$\begin{aligned}\frac{\partial E_x}{\partial z} &= j\beta E_x & \frac{\partial E_y}{\partial z} &= j\beta E_y \\ \frac{\partial H_x}{\partial z} &= j\beta H_x & \frac{\partial H_y}{\partial z} &= j\beta H_y\end{aligned}$$

we can equate the vector components on each side of the equation to write the six components of the electric and magnetic field as

$$j\omega\epsilon E_x = \frac{\partial H_z}{\partial y} + j\beta H_y \quad (1a)$$

$$j\omega\epsilon E_y = -j\beta H_x - \frac{\partial H_z}{\partial x} \quad (1b)$$

$$j\omega\epsilon E_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \quad (1c)$$

$$-j\omega\mu H_x = \frac{\partial E_z}{\partial y} + j\beta E_y \quad (2a)$$

$$-j\omega\mu H_y = -j\beta E_x - \frac{\partial E_z}{\partial x} \quad (2b)$$

$$-j\omega\mu H_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \quad (2c)$$

Equations (1) and (2) are valid for any wave (guided or unguided) propagating in the z -direction in a source-free region with a propagation constant of $j\beta$. We may use Equations (1) and (2) to solve for the longitudinal field components in terms of the transverse field components.

$$\text{solve (1a) and (2b) for } H_y \quad E_x = \frac{-j}{k_c^2} \left(\beta \frac{\partial E_z}{\partial x} + \omega \mu \frac{\partial H_z}{\partial y} \right)$$

$$\text{solve (1b) and (2a) for } H_x \quad E_y = \frac{j}{k_c^2} \left(-\beta \frac{\partial E_z}{\partial y} + \omega \mu \frac{\partial H_z}{\partial x} \right)$$

$$\text{solve (1b) and (2a) for } E_y \quad H_x = \frac{j}{k_c^2} \left(\omega \epsilon \frac{\partial E_z}{\partial y} - \beta \frac{\partial H_z}{\partial x} \right)$$

$$\text{solve (1a) and (2b) for } E_x \quad H_y = \frac{-j}{k_c^2} \left(\omega \epsilon \frac{\partial E_z}{\partial x} + \beta \frac{\partial H_z}{\partial y} \right)$$

where k_c is the cutoff wavenumber defined by

$$k_c^2 = k^2 - \beta^2$$

The cutoff wavenumber for the wave guiding structure is determined by the wavenumber of the insulating medium through which the wave propagates ($k = \omega \sqrt{\mu \epsilon}$) and the propagation constant for the structure ($j\beta$). The equations for the transverse components of the fields are valid for all of the modes defined previously. These transverse field component equations can be specialized for each one of these guided structure modes.

TEM Mode

Using the general equations for the transverse fields of guided waves [Equation (3)], we see that the transverse fields of a TEM mode (defined by $E_z = H_z = 0$) are non-zero only when $k_c = 0$. When the cutoff wavenumber of the TEM mode is zero, an indeterminate form of (0/0) results for each of the transverse field equations.

$$E_x = \frac{-j}{k_c^2} \left(\beta \frac{\partial E_z}{\partial x} + \omega \mu \frac{\partial H_z}{\partial y} \right) \quad (3a)$$

$$E_y = \frac{j}{k_c^2} \left(-\beta \frac{\partial E_z}{\partial y} + \omega \mu \frac{\partial H_z}{\partial x} \right) \quad (3b)$$

$$H_x = \frac{j}{k_c^2} \left(\omega \epsilon \frac{\partial E_z}{\partial y} - \beta \frac{\partial H_z}{\partial x} \right) \quad (3c)$$

$$H_y = \frac{-j}{k_c^2} \left(\omega \epsilon \frac{\partial E_z}{\partial x} + \beta \frac{\partial H_z}{\partial y} \right) \quad (3d)$$

A zero-valued cutoff wavenumber yields the following:

$$k_c^2 = k^2 - \beta^2 = 0 \quad \Rightarrow \quad \beta = k = \omega \sqrt{\mu \epsilon}$$

$$k_c = \omega_c \sqrt{\mu \epsilon} = 2\pi f_c \sqrt{\mu \epsilon} = 0 \quad \Rightarrow \quad f_c = 0$$

The first equation above shows that the phase constant β of the TEM mode on a guiding structure is equivalent to the phase constant of a plane wave propagating in a region characterized by the same medium between the conductors of the guiding structure. The second equation shows that the cutoff frequency of a TEM mode is 0 Hz. This means that TEM modes can be propagated at any non-zero frequency assuming the guiding structure can support a TEM mode.

Relationships between the transverse fields of the TEM mode can be determined by returning to the source-free Maxwell's equation results for guided waves [Equations (1) and (2)] and setting $E_z = H_z = 0$ and $\beta = k$.

$$E_x = \frac{k}{\omega \epsilon} H_y$$

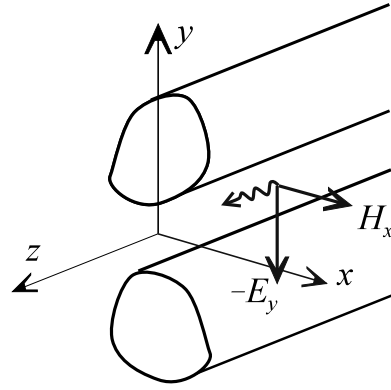
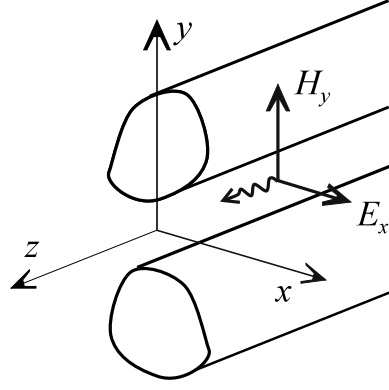
$$E_y = -\frac{k}{\omega \epsilon} H_x$$

$$\frac{\partial H_y}{\partial x} = \frac{\partial H_x}{\partial y}$$

$$H_x = -\frac{k}{\omega \mu} E_y$$

$$H_y = \frac{k}{\omega \mu} E_x$$

$$\frac{\partial E_y}{\partial x} = \frac{\partial E_x}{\partial y}$$



Note that the ratios of the TEM electric and magnetic field components define wave impedances which are equal to those of equivalent plane waves.

$$\frac{E_x}{H_y} = \frac{k}{\omega \epsilon} = \frac{\omega \mu}{k} = \sqrt{\frac{\mu}{\epsilon}} = \eta = Z_{TEM} \quad \Rightarrow \quad E_x = Z_{TEM} H_y$$

$$\frac{-E_y}{H_x} = \frac{k}{\omega \epsilon} = \frac{\omega \mu}{k} = \sqrt{\frac{\mu}{\epsilon}} = \eta = Z_{TEM} \quad \Rightarrow \quad E_y = -Z_{TEM} H_x$$

The previous results can be combined to yield

$$\mathbf{h}(x,y) = \frac{1}{Z_{TEM}} \mathbf{a}_z \times \mathbf{e}(x,y)$$

The fields of the TEM mode must also satisfy the respective wave equation:

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = \mathbf{0}$$

$$\nabla^2 \mathbf{H} + k^2 \mathbf{H} = \mathbf{0}$$

where

$$\mathbf{E}(x,y,z) = \mathbf{e}(x,y) e^{-j\beta z}$$

$$\mathbf{H}(x,y,z) = \mathbf{h}(x,y) e^{-j\beta z}$$

In rectangular coordinates, the vector Laplacian operator is

$$\begin{aligned} \nabla^2 \mathbf{F} &= \nabla^2 F_x \mathbf{a}_x + \nabla^2 F_y \mathbf{a}_y + \nabla^2 F_z \mathbf{a}_z \\ &= \left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_x}{\partial z^2} \right) \mathbf{a}_x \\ &\quad + \left(\frac{\partial^2 F_y}{\partial x^2} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_y}{\partial z^2} \right) \mathbf{a}_y \\ &\quad + \left(\frac{\partial^2 F_z}{\partial x^2} + \frac{\partial^2 F_z}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2} \right) \mathbf{a}_z \end{aligned}$$

By separating the rectangular coordinate components in the wave equation, we find that each of the field components $F \in (E_x, E_y, H_x, H_y)$ must then satisfy the same equation [Helmholtz equation].

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} + k^2 F = 0$$

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - \beta^2 F + k^2 F = 0$$

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + k_c^2 F = 0$$

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$$

so that the TEM field components must satisfy

$$\begin{aligned} \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} &= 0 & \frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} &= 0 \\ \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} &= 0 & \frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} &= 0 \end{aligned}$$

This result can be written in compact form as

$$\nabla_t^2 \mathbf{e}(x, y) = 0$$

$$\nabla_t^2 \mathbf{h}(x, y) = 0$$

where ∇_t^2 defines the transverse Laplacian operator which in rectangular coordinates is

$$\nabla_t^2 \mathbf{F} = \nabla^2 F_x \mathbf{a}_x + \nabla^2 F_y \mathbf{a}_y$$

According to the previous result, the transverse fields of the TEM mode must satisfy Laplace's equation with boundary conditions defined by the conductor geometry of the guiding structure, just like the static fields which would exist on the guiding structure for $f = 0$. Thus, the TEM transverse field vectors $\mathbf{e}(x,y)$ and $\mathbf{h}(x,y)$ are identical to the static fields for the transmission line. This allows us to solve for the static fields of a given guiding structure geometry (Laplace's equation) to determine the fields of the TEM mode.

TE Modes

The transverse fields of TE modes are found by simplifying the general guided wave equations in (3) with $E_z = 0$. The resulting transverse fields for TE modes are

$$E_x = \frac{-j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial y} \quad (4a)$$

$$E_y = \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial x} \quad (4b)$$

$$H_x = \frac{-j\beta}{k_c^2} \frac{\partial H_z}{\partial x} \quad (4c)$$

$$H_y = \frac{-j\beta}{k_c^2} \frac{\partial H_z}{\partial y} \quad (4d)$$

The cutoff wavenumber k_c must be non-zero to yield bounded solutions for the transverse field components of TE modes. This means that we must operate the guiding structure above the corresponding cutoff frequency for the particular TE mode to propagate. Note that all of the transverse field components of the TE modes can be determined once the single

longitudinal component (H_z) is found. The longitudinal field component H_z must satisfy the wave equation so that

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} + k^2 H_z = 0$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} - \beta^2 H_z + k^2 H_z = 0$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + k_c^2 H_z = 0$$

Given the basic form of the guided wave magnetic field

$$\mathbf{H}(x,y,z) = [\mathbf{h}(x,y) + h_z(x,y)\mathbf{a}_z] e^{-j\beta z}$$

$$H_z(x,y,z) = h_z(x,y) e^{-j\beta z}$$

we may write

$$\frac{\partial^2 h_z(x,y)}{\partial x^2} + \frac{\partial^2 h_z(x,y)}{\partial y^2} + k_c^2 h_z(x,y) = 0$$

The equation above represents a reduced Helmholtz equation which can be solved for $h_z(x,y)$ based on the boundary conditions of the guiding structure geometry. Once $h_z(x,y)$ is found, the longitudinal magnetic field is known, and all of the transverse field components are found by evaluating the derivatives in Equation (4).

The wave impedance for TE modes is found from Equation (4):

$$Z_{TE} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\omega\mu}{\beta} = \frac{k\eta}{\beta}$$

Note that the TE wave impedance is a function of frequency.

TM Modes

The transverse fields of TM modes are found by simplifying the general guided wave equations in (3) with $H_z = 0$. The resulting transverse fields for TM modes are

$$E_x = \frac{-j\beta}{k_c^2} \frac{\partial E_z}{\partial x} \quad (5a)$$

$$E_y = \frac{-j\beta}{k_c^2} \frac{\partial E_z}{\partial y} \quad (5b)$$

$$H_x = \frac{j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial y} \quad (5c)$$

$$H_y = \frac{-j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial x} \quad (5d)$$

The cutoff wavenumber k_c must also be non-zero to yield bounded solutions for the transverse field components of TM modes so that we must operate the guiding structure above the corresponding cutoff frequency for the particular TMmode to propagate. Note that all of the transverse field components of the TMmodes can be determined once the single longitudinal component (E_z) is found. The longitudinal field component E_z must satisfy the wave equation so that

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + k^2 E_z = 0$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - \beta^2 E_z + k^2 E_z = 0$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_c^2 E_z = 0$$

Given the basic form of the guided wave electric field

$$\mathbf{E}(x,y,z) = [\mathbf{e}(x,y) + e_z(x,y)\mathbf{a}_z] e^{-j\beta z}$$

$$E_z(x,y,z) = e_z(x,y) e^{-j\beta z}$$

we may write

$$\frac{\partial^2 e_z(x,y)}{\partial x^2} + \frac{\partial^2 e_z(x,y)}{\partial y^2} + k_c^2 e_z(x,y) = 0$$

The equation above represents a reduced Helmholtz equation which can be solved for $e_z(x,y)$ based on the boundary conditions of the guiding structure geometry. Once $e_z(x,y)$ is found, the longitudinal magnetic field is known, and all of the transverse field components are found by evaluating the derivatives in Equation (5).

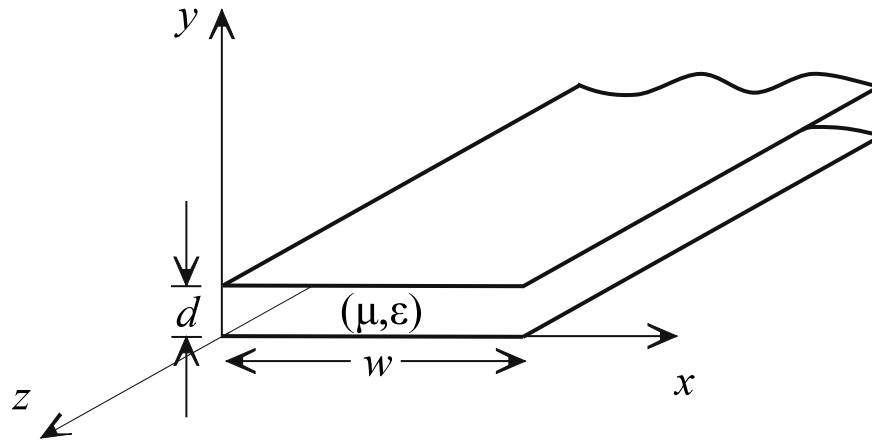
The wave impedance for TM modes is found from Equation (5):

$$Z_{TM} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\beta}{\omega\epsilon} = \frac{\beta\eta}{k}$$

Note that the TM wave impedance is also a function of frequency.

Parallel Plate Waveguide

The parallel plate waveguide is formed by two conducting plates of width w separated by a distance d as shown below. This “waveguide” can support TEM, TE and TM modes.

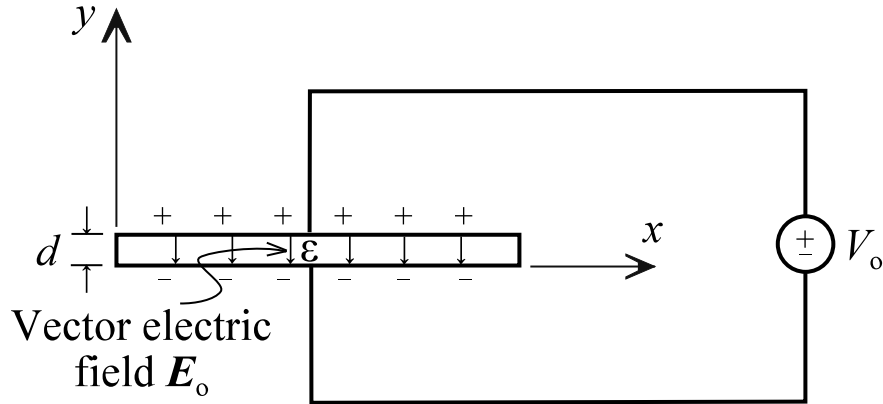


The following assumptions are made in the determination of the various modes on the parallel plate waveguide:

- (1) The waveguide is infinite in length (no reflections).
- (2) The waveguide conductors are PEC's and the dielectric is lossless.
- (3) The plate width is much larger than the plate separation ($w \gg d$) so that the variation of the fields with respect to x may be neglected.

Parallel Plate Waveguide TEM Mode

We have previously shown that the transverse fields of the TEM mode on a general wave guiding structure are equal to the corresponding static fields of the structure. The electrostatic field of the parallel plate waveguide ($w \gg d$) is equivalent to that found in the ideal parallel plate capacitor.



The vector electrostatic field in the ideal parallel plate capacitor is

$$\mathbf{E}_o = \frac{V_o}{d} (-\mathbf{a}_y)$$

Thus, the transverse electric field function for the TEM mode $\mathbf{e}(x,y)$ in the parallel plate waveguide is

$$\mathbf{e}(x,y) = \mathbf{E}_o = -\frac{V_o}{d} \mathbf{a}_y$$

Given $\beta=k$ for TEM waves, the overall electric field vector for the TEM mode on the parallel plate waveguide is

$$\mathbf{E}(x,y,z) = \mathbf{e}(x,y) e^{-j\beta z} = -\frac{V_o}{d} e^{-jkz} \mathbf{a}_y$$

A similar procedure could be followed for the magnetic field of the parallel plate waveguide TEM mode (determine the magnetostatic field within the parallel plate waveguide given oppositely directed DC currents in the two plates). However, we have already determined a simple relationship between the transverse electric and magnetic fields of a general TEM mode:

$$\mathbf{h}(x,y) = \frac{1}{Z_{TEM}} \mathbf{a}_z \times \mathbf{e}(x,y)$$

where

$$Z_{TEM} = \eta = \sqrt{\frac{\mu}{\epsilon}}$$

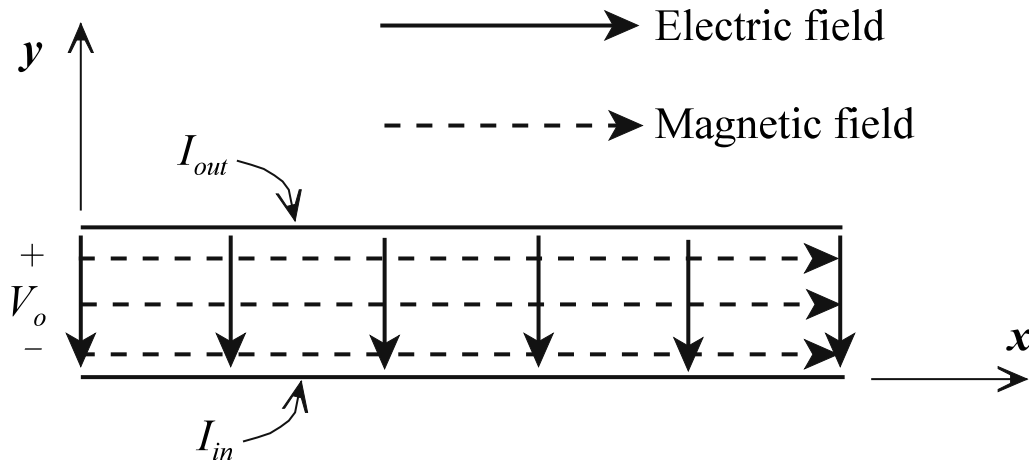
The transverse magnetic field function of the TEM mode becomes

$$\mathbf{h}(x,y) = \frac{1}{\eta} \mathbf{a}_z \times \left(-\frac{V_o}{d} \mathbf{a}_y \right) = \frac{V_o}{\eta d} \mathbf{a}_x$$

and the overall magnetic field vector is

$$\mathbf{H}(x,y,z) = \mathbf{h}(x,y) e^{-j\beta z} = \frac{V_o}{\eta d} e^{-jkz} \mathbf{a}_x$$

Note that both the electric field and magnetic field of the parallel plate waveguide TEM mode are independent of the transverse directions (x and y). Thus, these fields are uniform over the cross-section of the waveguide as shown below. The current directions in the two plates correspond to the direction of propagation assumed for the waves.



If we consider the parallel plate waveguide as a transmission line (carrying the TEM mode), the characteristic impedance Z_o of this line is defined as the ratio of voltage to current (for the respective forward and reverse traveling waves) at any point on the line:

$$Z_o = \frac{V_o^+}{I_o^+} = \frac{-V_o^-}{I_o^-}$$

For the infinite length line, we have only forward traveling waves. The voltage and current of these waves are defined according to

$$Z_o = \frac{V_o^+}{I_o^+} = \frac{-\int_L \mathbf{E} \cdot d\mathbf{l}}{\int_C \mathbf{H} \cdot d\mathbf{l}}$$

where the path L goes from the lower plate to the upper plate and the contour C encloses the upper conductor. The evaluation of the integrals yields

$$Z_o = \frac{V_o^+}{I_o^+} = \frac{-\int_0^d E_y dy}{\int_0^w H_x dx} = \frac{\frac{V_o}{d}(d)}{\frac{V_o}{\eta d}(w)} = \frac{\eta d}{w} \quad (\Omega)$$

The phase velocity of the TEM mode on the parallel plate waveguide is given by

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$$

Parallel Plate Waveguide TE Modes

The governing partial differential equation for the longitudinal magnetic field function of the TE mode on a general wave guiding structure is

$$\frac{\partial^2 h_z(x,y)}{\partial x^2} + \frac{\partial^2 h_z(x,y)}{\partial y^2} + k_c^2 h_z(x,y) = 0$$

where $k_c^2 = k^2 - \beta^2$. For the parallel plate waveguide with ($w \gg d$), we assume that the variation of the fields with respect to x is negligible which yields

$$\frac{\partial^2 h_z(x,y)}{\partial y^2} + k_c^2 h_z(x,y) = 0$$

The general solution to the equation for the longitudinal magnetic field function is

$$h_z(x,y) = A \sin k_c y + B \cos k_c y$$

such that the longitudinal magnetic field is given by

$$H_z(x,y,z) = [A \sin k_c y + B \cos k_c y] e^{-j\beta z}$$

The constants A and B are found by applying the appropriate boundary conditions for the longitudinal and transverse fields within the waveguide. For the parallel plate waveguide, with PEC's at $y = 0$ and $y = d$, the TE boundary conditions are

$$E_x(x,0,z) = 0$$

$$E_x(x,d,z) = 0$$

The x component of the TE mode electric field is related to H_z by

$$E_x = \frac{-j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial y} = \frac{-j\omega\mu}{k_c} [A \cos k_c y - B \sin k_c y] e^{-j\beta z}$$

Application of the TE boundary conditions gives

$$E_x(x, 0, z) = \frac{-j\omega\mu}{k_c} A e^{-j\beta z} = 0 \quad \Rightarrow \quad A = 0$$

$$E_x(x, d, z) = \frac{j\omega\mu}{k_c} [B \sin k_c d] e^{-j\beta z} = 0 \quad \Rightarrow \quad \begin{aligned} k_c d &= n\pi \\ n &= 1, 2, 3, \dots \end{aligned}$$

This yields

$$H_z(x, y, z) = B_n \cos \frac{n\pi y}{d} e^{-j\beta z}$$

$$E_x(x, y, z) = \frac{j\omega\mu}{k_c} B_n \sin \frac{n\pi y}{d} e^{-j\beta z}$$

where B_n is an amplitude constant associated with the discrete TE_n mode. The remaining transverse fields are related to H_z by

$$E_y = \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial x} = 0$$

$$H_x = \frac{-j\beta}{k_c^2} \frac{\partial H_z}{\partial x} = 0$$

$$H_y = \frac{-j\beta}{k_c^2} \frac{\partial H_z}{\partial y} = \frac{j\beta}{k_c} B_n \sin \frac{n\pi y}{d} e^{-j\beta z}$$

The propagation constant of the TE_n mode is

$$\beta = \sqrt{k^2 - \left(\frac{n\pi}{d} \right)^2}$$

The cutoff frequency for the TE_n mode is found according to the value of the propagation constant. Note that

$$\begin{array}{lll} \text{if } \frac{n\pi}{d} < k & \beta \text{ is real} & e^{-j\beta z} \text{ (propagates unattenuated)} \\ \text{if } \frac{n\pi}{d} > k & \beta \text{ is imaginary} & e^{-j\beta z} \text{ (attenuated)} \end{array}$$

These attenuated modes are called *evanescent* modes. The cutoff frequencies for the TE_n propagating modes are defined by

$$\begin{aligned} \frac{n\pi}{d} = k_c = \omega_c \sqrt{\mu\epsilon} = 2\pi f_c \sqrt{\mu\epsilon} \\ f_c = \frac{n\pi}{d2\pi\sqrt{\mu\epsilon}} = \frac{n}{2d\sqrt{\mu\epsilon}} \end{aligned}$$

The phase constant β may be expressed in terms of the cutoff frequency as

$$\begin{aligned} k_c^2 = k^2 - \beta^2 \quad \Rightarrow \quad \beta = \sqrt{k^2 - k_c^2} = k \sqrt{1 - \left(\frac{k_c}{k}\right)^2} \\ = k \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} = k \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \end{aligned}$$

From the equation for β in the parallel plate waveguide, we see that $\beta < k$. The wave impedance of the TE_n mode is

$$Z_{TE} = \frac{E_x}{H_y} = \frac{k\eta}{\beta} = \frac{k\eta}{k \sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = \frac{\eta}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

Parallel Plate Waveguide TM Modes

The governing partial differential equation for the longitudinal electric field function of the TM mode on a general wave guiding structure is

$$\frac{\partial^2 e_z(x,y)}{\partial x^2} + \frac{\partial^2 e_z(x,y)}{\partial y^2} + k_c^2 e_z(x,y) = 0$$

where $k_c^2 = k^2 - \beta^2$. For the parallel plate waveguide with ($w \gg d$), we assume that the variation of the fields with respect to x is negligible which yields

$$\frac{\partial^2 e_z(x,y)}{\partial y^2} + k_c^2 e_z(x,y) = 0$$

The general solution to the equation for the longitudinal electric field function is

$$e_z(x,y) = A \sin k_c y + B \cos k_c y$$

such that the longitudinal electric field is given by

$$E_z(x,y,z) = [A \sin k_c y + B \cos k_c y] e^{-j\beta z}$$

The constants A and B are found by applying the appropriate boundary conditions for the longitudinal and transverse fields within the waveguide. For the parallel plate waveguide, with PEC's at $y = 0$ and $y = d$, the TM boundary conditions are

$$E_x(x,0,z) = 0 \quad E_z(x,0,z) = 0$$

$$E_x(x,d,z) = 0 \quad E_z(x,d,z) = 0$$

The x component of the TM mode electric field is related to E_z by

$$E_x = \frac{-j\beta}{k_c^2} \frac{\partial E_z}{\partial y} = 0$$

Application of the TM boundary condition on E_z gives

$$E_z(x, 0, z) = B e^{-j\beta z} = 0 \quad \Rightarrow \quad B = 0$$

$$E_z(x, d, z) = A [\sin k_c d] e^{-j\beta z} = 0 \quad \Rightarrow \quad \begin{aligned} k_c d &= n\pi \\ n &= 1, 2, 3, \dots \end{aligned}$$

This yields

$$E_z(x, y, z) = A_n \sin \frac{n\pi y}{d} e^{-j\beta z}$$

where A_n is an amplitude constant associated with the discrete TM_n mode. The remaining transverse fields are related to E_z by

$$H_x = \frac{j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial y} = \frac{j\omega\epsilon}{k_c} A_n \cos \frac{n\pi y}{d} e^{-j\beta z}$$

$$H_y = \frac{-j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial x} = 0$$

$$E_y = \frac{-j\beta}{k_c^2} \frac{\partial E_z}{\partial y} = \frac{-j\beta}{k_c} A_n \cos \frac{n\pi y}{d} e^{-j\beta z}$$

Given that $k_c = n\pi/d$ is the same cutoff wavenumber found for the TE_n modes, the propagation constant and the cutoff frequency of the TM_n mode are the same as that for the TE_n mode.

$$f_c = \frac{n\pi}{d2\pi\sqrt{\mu\epsilon}} = \frac{n}{2d\sqrt{\mu\epsilon}}$$

$$\beta = \sqrt{k^2 - k_c^2} = k \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

Since the TE_n and TM_n modes have the same cutoff frequency (*degenerate modes*), we cannot propagate one mode without the other. Note that the fields of the parallel plate waveguide TM_n mode with $n = 0$ are identical to those of the TEM mode.

The wave impedance of the TM_n mode is

$$Z_{TM} = \frac{-E_y}{H_x} = \frac{\beta \eta}{k} = \frac{\eta k \sqrt{1 - \left(\frac{f_c}{f}\right)^2}}{k} = \eta \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

Waveguide Phase Velocity and Wavelength

The phase velocity for the TE_n and TM_n modes on the parallel plate waveguide is

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{k \sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = \frac{\frac{1}{\sqrt{\mu \epsilon}}}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

The phase velocity for propagating modes ($f > f_c$) is actually larger than the phase velocity of a plane wave propagating in a medium defined by (μ, ϵ) . In the case of the plane wave, the phase velocity (the speed at which the points of constant phase on the wave move) is equal to the velocity of the wave. For the waveguide, the phase velocity is not equal to the speed at which the overall wave propagates along the guide (this velocity is known as the group velocity and will be defined later). To physically interpret the phase velocity, we consider the fields of the parallel plate waveguide which vary as

$$\begin{aligned}
\sin \frac{n\pi y}{d} e^{-j\beta z} &= \left[\frac{e^{j\frac{n\pi y}{d}} - e^{-j\frac{n\pi y}{d}}}{2j} \right] e^{-j\beta z} \\
&= \frac{1}{2j} \left[e^{j\left(\frac{n\pi y}{d} - \beta z\right)} - e^{-j\left(\frac{n\pi y}{d} + \beta z\right)} \right] \\
\cos \frac{n\pi y}{d} e^{-j\beta z} &= \left[\frac{e^{j\frac{n\pi y}{d}} + e^{-j\frac{n\pi y}{d}}}{2} \right] e^{-j\beta z} \\
&= \frac{1}{2} \left[e^{j\left(\frac{n\pi y}{d} - \beta z\right)} + e^{-j\left(\frac{n\pi y}{d} + \beta z\right)} \right]
\end{aligned}$$

The overall waves within the parallel plate waveguide are mathematically equivalent to two plane waves propagating along the waveguide in at angles defined by the $-y$ and $+z$ components of the first terms and the $+y$ and $+z$ components of the second terms. Thus, the wave on the parallel plate waveguide looks like the superposition of two plane waves being reflected between the two plates as they propagate.

For propagating modes within a waveguide, the phase constant is related to the wavelength within the waveguide (λ_g) by

$$\beta = \frac{2\pi}{\lambda_g}$$

The guide wavelength is the distance between equiphase planes along the direction of propagation (z -axis). For the parallel plate waveguide, the guide wavelength is

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{k \sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = \frac{\lambda}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

where λ is the wavelength of a plane wave propagating in a medium defined by (μ, ϵ) . For propagating modes ($f > f_c$), the guide wavelength is longer than the corresponding plane wave wavelength.

We may also define a so-called *cutoff wavelength* using the plane wave equation

$$\lambda = \frac{2\pi}{k} \quad \Rightarrow \quad \lambda_c = \frac{2\pi}{k_c} = \frac{2\pi}{2\pi f_c \sqrt{\mu\epsilon}} = \frac{1}{f_c \sqrt{\mu\epsilon}}$$

The lowest frequency TE or TM mode (their cutoff frequencies are the same) are the $n = 1$ modes where

$$f_c(n = 1) = \frac{1}{2d\sqrt{\mu\epsilon}}$$

which gives a cutoff wavelength of

$$\lambda_c = 2d$$

for the TE₁ and TM₁ modes in a parallel plate waveguide.

Conductor and Dielectric Losses in a Waveguide

All of the previous waveguide equations were derived assuming the waveguide consists of perfect conductors and dielectrics. When losses are incorporated, the form of the z -dependent wave propagation terms must be modified accordingly:

$$e^{-j\beta z} \quad \Rightarrow \quad e^{-\gamma z}$$

where the complex propagation constant γ may be written as

$$\gamma = \alpha + j\beta = (\alpha_c + \alpha_d) + j\beta$$

where α_c and α_d account for the conductor and dielectric losses respectively.

To determine the conductor losses, we may employ the perturbation method which requires that we know the field distribution within the waveguide. Since different modes have different field distributions, each mode will have a different attenuation constant due to conductor losses.

If the waveguide is characterized by a homogeneous dielectric, then the attenuation constant due to dielectric losses can be determined without having to know the exact distribution of fields within the waveguide.

Dielectric Losses

If we account for dielectric losses only within the waveguide, the propagation constant becomes

$$\gamma = \alpha_c + j\beta = \sqrt{k_c^2 - k^2} = \sqrt{k_c^2 - \omega^2 \mu \epsilon}$$

If the magnetic loss in the dielectric is assumed to be negligible, then

$$\mu = \mu_o$$

$$\epsilon = \epsilon_r \epsilon_o (1 - \tan \delta)$$

The propagation constant can then be written as

$$\gamma = \sqrt{k_c^2 - k^2(1 - j \tan \delta)} = \sqrt{(k_c^2 - k^2) + j k^2 \tan \delta}$$

where $k^2 = \omega^2 \mu_o \epsilon_r \epsilon_o$ is the square of the real wavenumber for the dielectric without loss. The equation for the propagation constant can be rewritten as

$$\gamma = \sqrt{k_c^2 - k^2} \sqrt{1 + j \frac{k^2 \tan \delta}{(k_c^2 - k^2)}}$$

$$\sqrt{k_c^2 - k^2} = \sqrt{-(k^2 - k_c^2)} = \sqrt{-\beta^2} = j\beta \quad (\text{for TE, TM modes})$$

$$\gamma = j\beta \sqrt{1 - j \frac{k^2 \tan \delta}{\beta^2}}$$

If the dielectric losses are small (the loss tangent is small), then we may apply the following series approximation to the square root term in the propagation constant expression:

$$\sqrt{1-x} \approx 1 - \frac{x}{2} \quad (x \ll 1)$$

This yields

$$\gamma \approx j\beta \left[1 - j \frac{k^2 \tan \delta}{2\beta^2} \right] = \frac{k^2 \tan \delta}{2\beta} + j\beta = \alpha_d + j\beta$$

Thus, the attenuation constant due to dielectric losses for a TE or TM mode in any waveguide is found to be

$$\alpha_d = \frac{k^2 \tan \delta}{2\beta} \quad \left(\frac{\text{Np}}{\text{m}} \right) \quad (\text{TE and TM modes})$$

For the TEM mode, $\beta = k$ so that

$$\alpha_d = \frac{k \tan \delta}{2} \quad \left(\frac{\text{Np}}{\text{m}} \right) \quad (\text{TEM mode})$$

Conductor Losses

According to the perturbation method, the attenuation constant due to conductor losses is

$$\alpha_c = \frac{P_l}{2P_o}$$

where P_o is the power flow along the waveguide given by

$$P_o = \frac{1}{2} \text{Re} \int \int_{S_1} (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} \quad (\text{W/m}^2)$$

(S_l defines the cross-sectional surface of the waveguide) and where P_l is the power dissipated per unit length of the waveguide given by

$$P_l(z) = \left[\frac{R_s}{2} \int_{S_2} |J_s|^2 ds \right] / l = \left[\frac{R_s}{2} \int_{S_2} |H_t|^2 ds \right] / l$$

(S_2 defines the surface area of the waveguide for a unit length l).

Example (conductor loss in a parallel plate waveguide - TM modes)

The power flow along the parallel plate waveguide for a TM mode is given by

$$P_o = \frac{1}{2} \operatorname{Re} \iint_{S_1} (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} \quad d\mathbf{s} = dx dy \mathbf{a}_z \quad \begin{array}{l} (0 \leq x \leq w) \\ (0 \leq y \leq d) \end{array}$$

$$H_x = \frac{j\omega\epsilon}{k_c} A_n \cos \frac{n\pi y}{d} e^{-j\beta z}$$

$$E_y = \frac{-j\beta}{k_c} A_n \cos \frac{n\pi y}{d} e^{-j\beta z}$$

$$\begin{aligned} \mathbf{E} \times \mathbf{H}^* &= \left(\frac{-j\beta}{k_c} A_n \cos \frac{n\pi y}{d} e^{-j\beta z} \mathbf{a}_y \right) \times \left(\frac{-j\omega\epsilon}{k_c} A_n^* \cos \frac{n\pi y}{d} e^{j\beta z} \mathbf{a}_x \right) \\ &= \frac{\beta\omega\epsilon}{k_c^2} |A_n|^2 \cos^2 \frac{n\pi y}{d} \mathbf{a}_z \end{aligned}$$

$$P_o = \frac{\beta\omega\epsilon}{2k_c^2} |A_n|^2 \int_{y=0}^d \int_{x=0}^w \cos^2 \frac{n\pi y}{d} dx dy = \frac{\beta\omega\epsilon}{2k_c^2} |A_n|^2 (w) \begin{pmatrix} d & n=0 \\ \frac{d}{2} & n \geq 1 \end{pmatrix}$$

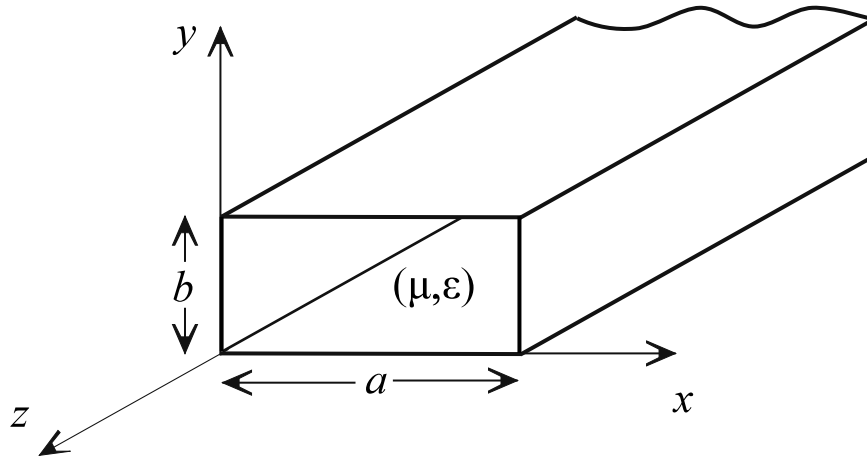
The power dissipated per unit length of the parallel plate waveguide TM mode is

$$\begin{aligned}
 P_l(z) &= \left[\frac{R_s}{2} \int_{S_2} |H_t|^2 ds \right] / l = (2) \frac{R_s}{2l} \int_{z=0}^l \int_{x=0}^w |H_x(y=0)|^2 dx dz \\
 &= R_s \int_{x=0}^w \frac{\omega^2 \epsilon^2}{k_c^2} |A_n|^2 dx = \frac{\omega^2 \epsilon^2 R_s |A_n|^2 w}{k_c^2}
 \end{aligned}$$

According to the perturbation method, the attenuation constant due to conductor losses for the parallel plate waveguide TM mode is

$$\begin{aligned}
 \alpha_c &= \frac{P_l}{2P_o} = \frac{\frac{\omega^2 \epsilon^2 R_s |A_n|^2 w}{k_c^2}}{2 \frac{\beta \omega \epsilon}{2 k_c^2} |A_n|^2 (w) \begin{pmatrix} d & n=0 \\ \frac{d}{2} & n \geq 1 \end{pmatrix}} \\
 &= \begin{pmatrix} \frac{\omega \epsilon R_s}{\beta d} & n=0 \\ \frac{2 \omega \epsilon R_s}{\beta d} & n \geq 1 \end{pmatrix}
 \end{aligned}$$

Rectangular Waveguide



The rectangular waveguide can support only TE and TM modes given only a single conductor. The rectangular cross-section ($a > b$) allows for single-mode operation.

Rectangular Waveguide TE modes

The longitudinal magnetic field function for the TE modes within the rectangular waveguide must satisfy

$$\frac{\partial^2 h_z(x,y)}{\partial x^2} + \frac{\partial^2 h_z(x,y)}{\partial y^2} + k_c^2 h_z(x,y) = 0$$

where $k_c^2 = k^2 - \beta^2$. The magnetic field function may be determined using the separation of variables technique by assuming a solution of the form

$$h_z(x,y) = X(x)Y(y)$$

Inserting the assumed solution into the governing partial differential equation yields

$$Y(y) \frac{d^2 X(x)}{dx^2} + X(x) \frac{d^2 Y(y)}{dy^2} + k_c^2 X(x)Y(y) = 0$$

Dividing by $X(x)Y(y)$ gives

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + k_c^2 = 0 \quad (1)$$

The first two terms in (1) are each dependent on only one variable. In order for (1) to be satisfied for every x and y within the waveguide, each of the first two terms must be equal to a constant.

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = -k_x^2 \quad \Rightarrow \quad \frac{d^2 X(x)}{dx^2} + k_x^2 X(x) = 0$$

$$\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = -k_y^2 \quad \Rightarrow \quad \frac{d^2 Y(y)}{dy^2} + k_y^2 Y(y) = 0$$

$$k_c^2 = k_x^2 + k_y^2 \quad (\text{separation equation})$$

The original second order partial differential equation dependent on two variables has been separated into two second order pure differential equations each dependent on only one variable. The general solutions to the two separate differential equations are

$$X(x) = A \sin k_x x + B \cos k_x x$$

$$Y(y) = C \sin k_y y + D \cos k_y y$$

The resulting longitudinal magnetic field function for the rectangular waveguide TE modes is

$$h_z(x, y) = X(x)Y(y) = (A \sin k_x x + B \cos k_x x)(C \sin k_y y + D \cos k_y y)$$

The longitudinal magnetic field is

$$H_z(x, y, z) = (A \sin k_x x + B \cos k_x x)(C \sin k_y y + D \cos k_y y) e^{-j\beta z}$$

The TE boundary conditions for the rectangular waveguide are

$$E_x(x, 0, z) = E_x(x, b, z) = 0 \quad (\text{horizontal walls})$$

$$E_y(0, y, z) = E_y(a, y, z) = 0 \quad (\text{vertical walls})$$

where

$$\begin{aligned} E_x &= \frac{-j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial y} \\ &= \frac{-j\omega\mu}{k_c^2} k_y (A \sin k_x x + B \cos k_x x) (C \cos k_y y - D \sin k_y y) e^{-j\beta z} \\ E_y &= \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial x} \\ &= \frac{j\omega\mu}{k_c^2} k_x (A \cos k_x x - B \sin k_x x) (C \sin k_y y + D \cos k_y y) e^{-j\beta z} \end{aligned}$$

The application of the boundary conditions yields

$$E_y(0, y, z) = 0 \quad \Rightarrow \quad A = 0$$

$$E_y(a, y, z) = 0 \quad \Rightarrow \quad k_x a = m\pi \quad (m = 0, 1, 2, \dots)$$

$$E_x(x, 0, z) = 0 \quad \Rightarrow \quad C = 0$$

$$E_x(x, b, z) = 0 \quad \Rightarrow \quad k_y b = n\pi \quad (n = 0, 1, 2, \dots)$$

The resulting product of the constants B and D into combined into one constant (A_{mn}).

$$H_z(x, y, z) = A_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$$

$$E_x(x,y,z) = \frac{j\omega\mu n\pi}{k_c^2 b} A_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$$

$$E_y(x,y,z) = -\frac{j\omega\mu m\pi}{k_c^2 a} A_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$$

The transverse components of the magnetic field are

$$H_x(x,y,z) = -\frac{j\beta}{k_c^2} \frac{\partial H_z}{\partial x} = \frac{j\beta m\pi}{k_c^2 a} A_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$$

$$H_y(x,y,z) = -\frac{j\beta}{k_c^2} \frac{\partial H_z}{\partial y} = \frac{j\beta n\pi}{k_c^2 b} A_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$$

The index designation for the discrete TE modes is TE_{mn} . Note that the case of $n = m = 0$ is not allowed since this would make all of the transverse field components zero.

The phase constant and the cutoff wavenumber of the TE_{mn} mode are

$$\beta_{mn} = \sqrt{k^2 - k_{c_{mn}}^2} = \sqrt{k^2 - (k_x^2 + k_y^2)}$$

$$k_{c_{mn}} = \sqrt{k_x^2 + k_y^2} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

The phase constant is real if $k > k_{c_{mn}}$ (unattenuated propagating modes) and imaginary if $k < k_{c_{mn}}$ (evanescent modes). The cutoff frequency for the TE_{mn} mode is defined by $k = k_{c_{mn}}$ which gives

$$k_{c_{mn}} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = 2\pi f_{c_{mn}} \sqrt{\mu\epsilon}$$

$$f_{c_{mn}} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Note that we may again write the phase constant in terms of the corresponding cutoff frequency

$$\beta_{mn} = k \sqrt{1 - \left(\frac{k_{c_{mn}}}{k}\right)^2} = k \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}$$

The wave impedance of the TE_{mn} mode is

$$Z_{TE_{mn}} = \frac{k\eta}{\beta} = \frac{\eta}{\sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

and the guide wavelength is

$$\lambda_{g_{mn}} = \frac{2\pi}{\beta_{mn}} = \frac{\lambda}{\sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

The cutoff wavelength is given by

$$\lambda_{c_{mn}} = \frac{2\pi}{k_{c_{mn}}} = \frac{1}{f_{c_{mn}} \sqrt{\mu\epsilon}}$$

Rectangular Waveguide TM modes

The longitudinal electric field function for the TM modes within the rectangular waveguide must satisfy

$$\frac{\partial^2 e_z(x,y)}{\partial x^2} + \frac{\partial^2 e_z(x,y)}{\partial y^2} + k_c^2 e_z(x,y) = 0$$

where $k_c^2 = k^2 - \beta^2$. Note that this is the same partial differential equation that the longitudinal magnetic field satisfies in the TE case. Thus, the TM magnetic field function may be determined using the same technique used in the TE case (separation of variables) by assuming a solution of the form

$$e_z(x,y) = X(x)Y(y)$$

The resulting differential equations for the component functions are

$$\frac{d^2 X(x)}{dx^2} + k_x^2 X(x) = 0$$

$$\frac{d^2 Y(y)}{dy^2} + k_y^2 Y(y) = 0$$

$$k_c^2 = k_x^2 + k_y^2$$

with solutions of the form

$$X(x) = A \sin k_x x + B \cos k_x x$$

$$Y(y) = C \sin k_y y + D \cos k_y y$$

The resulting longitudinal electric field function for the rectangular waveguide TM modes is

$$e_z(x,y) = X(x)Y(y) = (A \sin k_x x + B \cos k_x x)(C \sin k_y y + D \cos k_y y)$$

The longitudinal electric field is

$$E_z(x,y,z) = (A \sin k_x x + B \cos k_x x)(C \sin k_y y + D \cos k_y y) e^{-j\beta z}$$

The TM boundary conditions for the rectangular waveguide are

$$E_x(x,0,z) = E_x(x,b,z) = 0 \quad (\text{horizontal walls})$$

$$E_z(x,0,z) = E_z(x,b,z) = 0 \quad (\text{horizontal walls})$$

$$E_y(0,y,z) = E_y(a,y,z) = 0 \quad (\text{vertical walls})$$

$$E_z(0,y,z) = E_z(a,y,z) = 0 \quad (\text{vertical walls})$$

We will find that satisfaction of the boundary conditions on the longitudinal electric field automatically satisfies the boundary conditions on the transverse fields.

$$E_z(0,y,z) = 0 \quad \Rightarrow \quad B = 0$$

$$E_z(a,y,z) = 0 \quad \Rightarrow \quad k_x a = m\pi \quad (m = 1, 2, \dots)$$

$$E_z(x,0,z) = 0 \quad \Rightarrow \quad D = 0$$

$$E_z(x,b,z) = 0 \quad \Rightarrow \quad k_y b = n\pi \quad (n = 1, 2, \dots)$$

Note that the values of k_x and k_y are identical to those of the corresponding TE modes. The resulting product of the constants A and C into combined into one constant (B_{mn}).

$$E_z(x,y,z) = B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$$

The resulting transverse fields are

$$E_x(x,y,z) = -\frac{j\beta}{k_c^2} \frac{\partial E_z}{\partial x} = -\frac{j\beta m\pi}{k_c^2 a} B_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$$

$$E_y(x,y,z) = -\frac{j\beta}{k_c^2} \frac{\partial E_z}{\partial y} = -\frac{j\beta n\pi}{k_c^2 b} B_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$$

$$H_x(x,y,z) = \frac{j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial y} = \frac{j\omega\epsilon n\pi}{k_c^2 b} B_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$$

$$H_y(x,y,z) = -\frac{j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial x} = -\frac{j\omega\epsilon m\pi}{k_c^2 a} B_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$$

The phase constant, cutoff wavenumber, cutoff frequency, guide wavelength, and cutoff guide wavelength for the TM_{mn} mode are all identical to those of the corresponding TE_{mn} mode.

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = k \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}$$

$$k_{c_{mn}} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = 2\pi f_{c_{mn}} \sqrt{\mu\epsilon}$$

$$f_{c_{mn}} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\lambda_{g_{mn}} = \frac{2\pi}{\beta_{mn}} = \frac{\lambda}{\sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

$$\lambda_{c_{mn}} = \frac{2\pi}{k_{c_{mn}}} = \frac{1}{f_{c_{mn}} \sqrt{\mu\epsilon}}$$

The wave impedance of the TM_{mn} mode is

$$Z_{TM_{mn}} = \frac{\beta\eta}{k} = \eta \sqrt{1 - \left(\frac{f_{c_{mn}}}{f} \right)^2}$$

Rectangular Waveguide Modes

The following TE and TM modes can propagate in a rectangular waveguide:

$$TE_{mn} \quad n = 0, 1, 2, \dots \quad m = 0, 1, 2, \dots \quad (m = n \neq 0)$$

$$TM_{mn} \quad n = 1, 2, 3, \dots \quad m = 1, 2, 3, \dots$$

The cutoff frequencies for these modes are defined by

$$f_{c_{mn}} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2}$$

According to the cutoff frequency equation, the cutoff frequencies of both the TE_{10} and TE_{01} modes are less than that of the lowest order TM mode (TM_{11}). Given $a > b$ for the rectangular waveguide, the TE_{10} has the lowest cutoff frequency of any of the rectangular waveguide modes and is thus the dominant mode. Note that the TE_{10} and TE_{01} modes are degenerate modes for a square waveguide. The rectangular waveguide allows one to operate at a frequency above the cutoff of the dominant TE_{10} mode but below that of the next highest mode to achieve single mode operation. A waveguide operating at a frequency where more than one mode propagates is said to be *overmoded*.

General Equations for Wave Guiding Structures in Cylindrical Coordinates

The general equations for the TEM, TE and TM modes of cylindrical wave guiding structures (circular waveguide, coaxial transmission line, etc.) should be defined in cylindrical coordinates. Assuming the axis of the guiding structure lies along the z -axis, the general expressions for the cylindrical coordinate fields may be written as

$$\mathbf{E}(\rho, \phi, z) = [\mathbf{e}(\rho, \phi) + e_z(\rho, \phi)\mathbf{a}_z] e^{-j\beta z}$$

$$\mathbf{H}(\rho, \phi, z) = [\mathbf{h}(\rho, \phi) + h_z(\rho, \phi)\mathbf{a}_z] e^{-j\beta z}$$

where the vectors $\mathbf{e}(\rho, \phi)$ and $\mathbf{h}(\rho, \phi)$ represent the transverse field components of the wave while the vectors $e_z(\rho, \phi)\mathbf{a}_z$ and $h_z(\rho, \phi)\mathbf{a}_z$ are the longitudinal components of the wave. Inserting the field expressions into the source-free Maxwell's equations, expanding the curl operators in cylindrical coordinates, and equating components yields

$$j\omega\epsilon E_\rho = \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} + j\beta H_\phi \quad (1a)$$

$$j\omega\epsilon E_\phi = -j\beta H_\rho - \frac{\partial H_z}{\partial \rho} \quad (1b)$$

$$j\omega\epsilon E_z = \frac{1}{\rho} \left[\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi} \right] \quad (1c)$$

$$-j\omega\mu H_\rho = \frac{1}{\rho} \frac{\partial E_z}{\partial \phi} + j\beta E_\phi \quad (2a)$$

$$-j\omega\mu H_\phi = -j\beta E_\rho - \frac{\partial E_z}{\partial \rho} \quad (2b)$$

$$-j\omega\mu H_z = \frac{1}{\rho} \left[\frac{\partial(\rho E_\phi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi} \right] \quad (2c)$$

Equations (1) and (2) may be used to solve for the longitudinal field components in terms of the transverse field components.

$$\text{solve (1a) and (2b) for } H_{\phi} \quad E_{\rho} = \frac{-j}{k_c^2} \left(\beta \frac{\partial E_z}{\partial \rho} + \frac{\omega \mu}{\rho} \frac{\partial H_z}{\partial \phi} \right)$$

$$\text{solve (1b) and (2a) for } H_{\rho} \quad E_{\phi} = \frac{-j}{k_c^2} \left(\frac{\beta}{\rho} \frac{\partial E_z}{\partial \phi} - \omega \mu \frac{\partial H_z}{\partial \rho} \right)$$

$$\text{solve (1b) and (2a) for } E_{\phi} \quad H_{\rho} = \frac{j}{k_c^2} \left(\frac{\omega \epsilon}{\rho} \frac{\partial E_z}{\partial \phi} - \beta \frac{\partial H_z}{\partial \rho} \right)$$

$$\text{solve (1a) and (2b) for } E_{\rho} \quad H_{\phi} = \frac{-j}{k_c^2} \left(\omega \epsilon \frac{\partial E_z}{\partial \rho} + \frac{\beta}{\rho} \frac{\partial H_z}{\partial \phi} \right)$$

where

$$k_c^2 = k^2 - \beta^2$$

TEM Mode

The TEM mode field relationships are found by inserting $E_z = H_z = 0$ and $\beta = k$ into Equations (1) and (2).

$$E_{\rho} = \frac{k}{\omega \epsilon} H_{\phi}$$

$$H_{\rho} = -\frac{k}{\omega \mu} E_{\phi}$$

$$E_{\phi} = -\frac{k}{\omega \epsilon} H_{\rho}$$

$$H_{\phi} = \frac{k}{\omega \mu} E_{\rho}$$

$$\frac{\partial(\rho H_{\phi})}{\partial \rho} = \frac{\partial H_{\rho}}{\partial \phi}$$

$$\frac{\partial(\rho E_{\phi})}{\partial \rho} = \frac{\partial E_{\rho}}{\partial \phi}$$

TE Modes ($E_z = 0$ in the general transverse field expressions)

$$E_\rho = \frac{-j\omega\mu}{k_c^2\rho} \frac{\partial H_z}{\partial\phi}$$

$$E_\phi = \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial\rho}$$

$$H_\rho = \frac{-j\beta}{k_c^2} \frac{\partial H_z}{\partial\rho}$$

$$H_\phi = \frac{-j\beta}{k_c^2\rho} \frac{\partial H_z}{\partial\phi}$$

TM Modes ($H_z = 0$ in the general transverse field expressions)

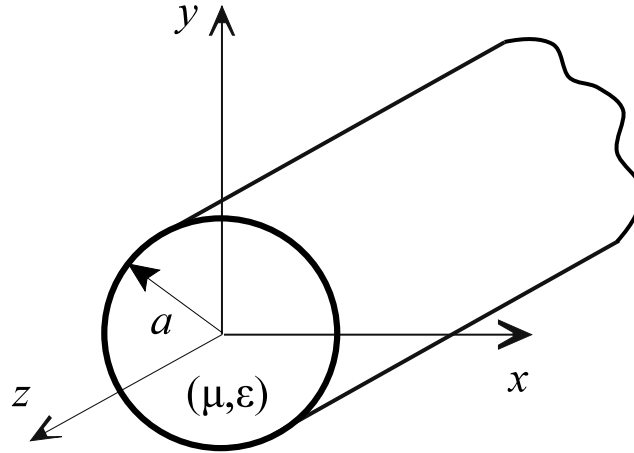
$$E_\rho = \frac{-j\beta}{k_c^2} \frac{\partial E_z}{\partial\rho}$$

$$E_\phi = \frac{-j\beta}{k_c^2\rho} \frac{\partial E_z}{\partial\phi}$$

$$H_\rho = \frac{j\omega\varepsilon}{k_c^2\rho} \frac{\partial E_z}{\partial\phi}$$

$$H_\phi = \frac{-j\omega\varepsilon}{k_c^2} \frac{\partial E_z}{\partial\rho}$$

Cylindrical Waveguide



The cylindrical waveguide can support only TE and TM modes given only a single conductor.

Cylindrical Waveguide TE modes

The longitudinal magnetic field of the TE modes within the cylindrical waveguide must satisfy

$$\nabla^2 H_z + k^2 H_z = 0$$

where

$$H_z(\rho, \phi, z) = h_z(\rho, \phi) e^{-j\beta z}$$

Inserting the expression for H_z into the differential equation yields

$$\frac{\partial^2 h_z(\rho, \phi)}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial h_z(\rho, \phi)}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 h_z(\rho, \phi)}{\partial \phi^2} + k_c^2 h_z(\rho, \phi) = 0$$

where $k_c^2 = k^2 - \beta^2$. The magnetic field function may be determined using the separation of variables technique by assuming a solution of the form

$$h_z(\rho, \phi) = R(\rho)P(\phi)$$

Inserting the assumed solution into the governing partial differential equation yields

$$P(\varphi) \frac{d^2 R(\rho)}{d\rho^2} + \frac{P(\varphi)}{\rho} \frac{dR(\rho)}{d\rho} + \frac{R(\rho)}{\rho^2} \frac{d^2 P(\varphi)}{d\varphi^2} + k_c^2 R(\rho) P(\varphi) = 0$$

Dividing by $R(\rho)P(\varphi)$ gives

$$\frac{1}{R(\rho)} \frac{d^2 R(\rho)}{d\rho^2} + \frac{1}{\rho R(\rho)} \frac{dR(\rho)}{d\rho} + \frac{1}{\rho^2 P(\varphi)} \frac{d^2 P(\varphi)}{d\varphi^2} + k_c^2 = 0$$

We multiply the expression above by ρ^2 in order to make the third term dependent on φ only. The result is

$$\frac{\rho^2}{R(\rho)} \frac{d^2 R(\rho)}{d\rho^2} + \frac{\rho}{R(\rho)} \frac{dR(\rho)}{d\rho} + \frac{1}{P(\varphi)} \frac{d^2 P(\varphi)}{d\varphi^2} + \rho^2 k_c^2 = 0 \quad (1)$$

According to the separation of variables technique, we may set the φ -dependent term in (1) equal to a constant ($-k_\varphi^2$). The resulting differential equation defining $P(\varphi)$ is

$$\frac{d^2 P(\varphi)}{d\varphi^2} + k_\varphi^2 P(\varphi) = 0$$

which has the general solution of

$$P(\varphi) = A \sin k_\varphi \varphi + B \cos k_\varphi \varphi$$

The function $P(\varphi)$ must be periodic in φ so that k_φ must be an integer (n).

$$P(\varphi) = A \sin n\varphi + B \cos n\varphi$$

Replacing the third term in (1) with $-k_\phi^2 = -n^2$ gives

$$\frac{\rho^2}{R(\rho)} \frac{d^2 R(\rho)}{d\rho^2} + \frac{\rho}{R(\rho)} \frac{dR(\rho)}{d\rho} + (\rho^2 k_c^2 - n^2) = 0 \quad (2)$$

Equation (3) is known as *Bessel's equation* which has solutions known as *Bessel functions*. We may write the general solution to Bessel's equation as

$$R(\rho) = C J_n(k_c \rho) + D Y_n(k_c \rho)$$

where

$J_n(k_c \rho)$ - nth order Bessel function of the first kind (argument = $k_c \rho$)

$Y_n(k_c \rho)$ - nth order Bessel function of the second kind (argument = $k_c \rho$)

The Bessel function of the second kind approaches ∞ as its argument approaches zero. Since the circular waveguide fields must be bounded at the origin ($\rho = 0$), then the constant D must be zero. The resulting longitudinal magnetic field function for the cylindrical waveguide TE modes is

$$h_z(\rho, \phi) = R(\rho)P(\phi) = J_n(k_c \rho)[A \sin n\phi + B \cos n\phi]$$

The longitudinal magnetic field is

$$H_z(\rho, \phi, z) = J_n(k_c \rho)[A \sin n\phi + B \cos n\phi] e^{-j\beta z}$$

Since there is no longitudinal electric field for the TE modes, the only TE boundary condition for the cylindrical waveguide is

$$E_\phi(a, \phi, z) = 0$$

where

$$E_{\phi} = \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial \rho} = \frac{j\omega\mu}{k_c^2} \frac{\partial}{\partial \rho} [J_n(k_c \rho)] (A \sin n\phi + B \cos n\phi) e^{-j\beta z}$$

Using the chain rule, the partial derivative of the Bessel function may be written as

$$\frac{\partial}{\partial \rho} [J_n(k_c \rho)] = \frac{\partial(k_c \rho)}{\partial \rho} \frac{\partial [J_n(k_c \rho)]}{\partial (k_c \rho)} = k_c J_n'(k_c \rho)$$

where $J_n'(k_c \rho)$ denotes the derivative with respect to the Bessel function argument. The TE boundary condition becomes

$$E_{\phi}(a, \rho, z) = \frac{j\omega\mu}{k_c} J_n'(k_c a) (A \sin n\phi + B \cos n\phi) e^{-j\beta z} = 0$$

Thus, the TE modes of the cylindrical waveguide are defined by

$$J_n'(k_c a) = 0$$

If we define the m th zero of derivative of the n th order Bessel function as p_{nm}' , then the TE _{nm} mode cutoff wavenumber is found by

$$k_{c_{nm}} a = p_{nm}' \quad \Rightarrow \quad k_{c_{nm}} = \frac{p_{nm}'}{a}$$

The resulting transverse fields of the TE _{nm} modes are

$$\begin{aligned} E_{\rho}(\rho, \phi, z) &= \frac{-j\omega\mu}{k_c^2 \rho} \frac{\partial H_z}{\partial \phi} \\ &= \frac{-j\omega\mu n}{k_c^2 \rho} J_n(k_c \rho) [A \cos n\phi - B \sin n\phi] e^{-j\beta z} \end{aligned}$$

$$\begin{aligned}
E_{\varphi}(\rho, \varphi, z) &= \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial \rho} \\
&= \frac{j\omega\mu}{k_c} J'_n(k_c \rho) [A \sin n\varphi + B \cos n\varphi] e^{-j\beta z} \\
H_{\rho}(\rho, \varphi, z) &= -\frac{j\beta}{k_c^2} \frac{\partial H_z}{\partial \rho} \\
&= -\frac{j\beta}{k_c} J'_n(k_c \rho) [A \sin n\varphi + B \cos n\varphi] e^{-j\beta z} \\
H_{\varphi}(\rho, \varphi, z) &= -\frac{j\beta}{k_c^2 \rho} \frac{\partial H_z}{\partial \varphi} \\
&= -\frac{j\beta n}{k_c^2 \rho} J_n(k_c \rho) [A \cos n\varphi - B \sin n\varphi] e^{-j\beta z}
\end{aligned}$$

The cutoff frequency of the TE_{nm} mode is given by

$$\begin{aligned}
k_{c_{nm}} &= 2\pi f_{c_{nm}} \sqrt{\mu\epsilon} = \frac{p'_{nm}}{a} \\
f_{c_{nm}} &= \frac{p'_{nm}}{2\pi a \sqrt{\mu\epsilon}}
\end{aligned}$$

The wavenumber for the TE_{nm} mode is

$$\beta_{nm} = \sqrt{k^2 - k_{c_{nm}}^2} = \sqrt{k^2 - \left(\frac{p'_{nm}}{a}\right)^2} = k \sqrt{1 - \left(\frac{f_{c_{nm}}}{f}\right)^2}$$

The wave impedance of the TE_{nm} mode is

$$Z_{TE_{nm}} = \frac{E_\rho}{H_\phi} = \frac{k\eta}{\beta} = \frac{\eta}{\sqrt{1 - \left(\frac{f_{c_{nm}}}{f}\right)^2}}$$

Cylindrical Waveguide TM modes

The longitudinal electric field of the TM modes within the cylindrical waveguide must satisfy

$$\nabla^2 E_z + k^2 E_z = 0$$

where

$$E_z(\rho, \phi, z) = e_z(\rho, \phi) e^{-j\beta z}$$

The longitudinal electric field function of the TM modes satisfies the same differential equation as the magnetic field of the TE modes. Thus, we may write the solution for $e_z(\rho, \phi)$ as

$$e_z(\rho, \phi) = R(\rho)P(\phi) = J_n(k_c \rho) [A \sin n\phi + B \cos n\phi]$$

The longitudinal electric field is

$$E_z(\rho, \phi, z) = J_n(k_c \rho) [A \sin n\phi + B \cos n\phi] e^{-j\beta z}$$

The TM boundary conditions for the cylindrical waveguide are

$$E_\phi(a, \phi, z) = 0$$

$$E_z(a, \phi, z) = 0$$

Just as in the case of the rectangular waveguide TM modes, we find that enforcement of the boundary condition on E_z automatically satisfies transverse field boundary condition. Application of the boundary condition on E_z yields

$$E_z(\rho, \phi, z) = [J_n(k_c a)] (A \sin n\phi + B \cos n\phi) e^{-j\beta z} = 0$$

Thus, the TM modes of the cylindrical waveguide are defined by

$$J_n(k_c a) = 0$$

If we define the m th zero of the n th order Bessel function as p_{nm} , then the TM_{nm} mode cutoff wavenumber is found by

$$k_{c_{nm}} a = p_{nm} \quad \Rightarrow \quad k_{c_{nm}} = \frac{p_{nm}}{a}$$

The resulting transverse fields of the TM_{nm} modes are

$$\begin{aligned} E_\rho(\rho, \phi, z) &= \frac{-j\beta}{k_c^2} \frac{\partial E_z}{\partial \rho} \\ &= \frac{-j\beta}{k_c} J'_n(k_c \rho) [A \sin n\phi + B \cos n\phi] e^{-j\beta z} \\ E_\phi(\rho, \phi, z) &= \frac{-j\beta}{k_c^2 \rho} \frac{\partial E_z}{\partial \phi} \\ &= \frac{-j\beta n}{k_c^2 \rho} J_n(k_c \rho) [A \cos n\phi - B \sin n\phi] e^{-j\beta z} \end{aligned}$$

$$\begin{aligned}
H_\rho(\rho, \phi, z) &= \frac{j\omega\epsilon}{k_c^2 \rho} \frac{\partial E_z}{\partial \phi} \\
&= \frac{j\omega\epsilon n}{k_c^2 \rho} J_n(k_c \rho) [A \cos n\phi - B \sin n\phi] e^{-j\beta z} \\
H_\phi(\rho, \phi, z) &= \frac{-j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial \rho} \\
&= \frac{-j\omega\epsilon}{k_c} J'_n(k_c \rho) [A \sin n\phi + B \cos n\phi] e^{-j\beta z}
\end{aligned}$$

The cutoff frequency of the TM_{nm} mode is given by

$$k_{c_{nm}} = 2\pi f_{c_{nm}} \sqrt{\mu\epsilon} = \frac{p_{nm}}{a}$$

$$f_{c_{nm}} = \frac{p_{nm}}{2\pi a \sqrt{\mu\epsilon}}$$

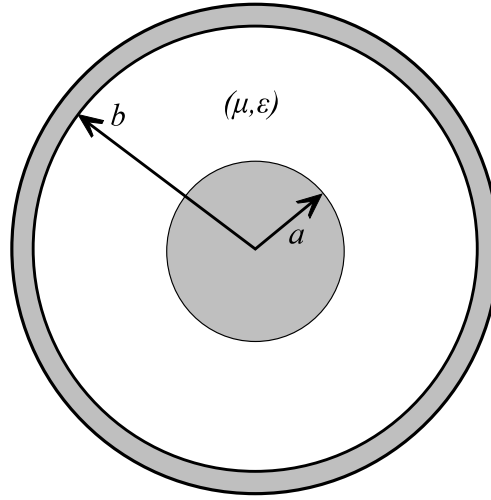
The wavenumber for the TM_{nm} mode is

$$\beta_{nm} = \sqrt{k^2 - k_{c_{nm}}^2} = \sqrt{k^2 - \left(\frac{p_{nm}}{a}\right)^2} = k \sqrt{1 - \left(\frac{f_{c_{nm}}}{f}\right)^2}$$

The wave impedance of the TM_{nm} mode is

$$Z_{TM_{nm}} = \frac{E_\rho}{H_\phi} = \frac{\eta\beta}{k} = \eta \sqrt{1 - \left(\frac{f_{c_{nm}}}{f}\right)^2}$$

Coaxial Transmission Line



The coaxial transmission line can support TEM, TE and TM modes. However, it is normally operated at frequencies where only the TEM mode (transmission line mode) propagates.

Coaxial Line TEM Modes

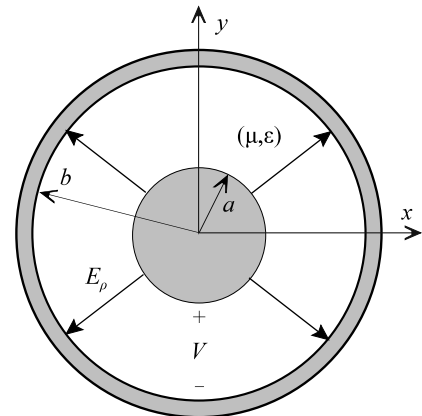
As previously illustrated in the parallel plate waveguide example, the transverse fields of the TEM mode may be defined as

$$\mathbf{E}(\rho, \phi, z) = \mathbf{e}(\rho, \phi) e^{-j\beta z}$$

$$\mathbf{H}(\rho, \phi, z) = \mathbf{h}(\rho, \phi) e^{-j\beta z}$$

where $\mathbf{e}(\rho, \phi)$ and $\mathbf{h}(\rho, \phi)$ are equivalent to the electrostatic and magnetostatic fields for the coaxial geometry. The electrostatic field is that of a coaxial capacitor with a voltage of V_o between the cylindrical conductors.

$$\mathbf{e}(\rho, \phi) = \mathbf{E}_o(\rho) = \frac{V_o}{\rho \ln(b/a)} \mathbf{a}_\rho$$

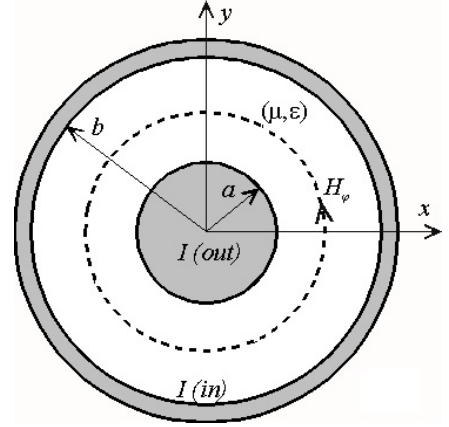


The magnetostatic field is that of a coaxial transmission line carrying DC currents in opposite directions. For the magnetostatic field to be the static limit of a +z directed wave on the transmission line, the current in the inner conductor must be outward.

$$\mathbf{h}(\rho, \varphi) = \frac{1}{Z_{TEM}} \mathbf{a}_z \times \mathbf{e}(\rho, \varphi)$$

$$Z_{TEM} = \eta = \sqrt{\frac{\mu}{\epsilon}}$$

$$\mathbf{h}(\rho, \varphi) = \mathbf{H}_o(\rho) = \frac{V_o}{\eta \rho \ln(b/a)} \mathbf{a}_\varphi$$



The TEM fields within the coaxial transmission line are

$$\mathbf{E}(\rho, \varphi, z) = \mathbf{e}(\rho, \varphi) e^{-j\beta z} = \frac{V_o e^{-j\beta z}}{\rho \ln(b/a)} \mathbf{a}_\rho$$

$$\mathbf{H}(\rho, \varphi, z) = \mathbf{h}(\rho, \varphi) e^{-j\beta z} = \frac{V_o e^{-j\beta z}}{\eta \rho \ln(b/a)} \mathbf{a}_\varphi$$

The TEM mode represents the dominant mode in a coaxial line with a cutoff frequency of zero. However, higher order TE and TM modes can propagate in the coaxial line at sufficiently high frequencies.

Coaxial Line TE Modes

The longitudinal magnetic field of the TE modes within the coaxial transmission line must satisfy

$$\nabla^2 H_z + k^2 H_z = 0$$

where

$$H_z(\rho, \phi, z) = h_z(\rho, \phi) e^{-j\beta z}$$

which leads to the same general separation of variables solution found for the circular waveguide.

$$h_z(\rho, \phi) = R(\rho)P(\phi)$$

$$R(\rho) = C J_n(k_c \rho) + D Y_n(k_c \rho)$$

$$P(\phi) = A \sin n\phi + B \cos n\phi$$

Note that the ρ -dependent term includes both Bessel functions of the first kind and second kind. We cannot eliminate the Bessel function of the second kind from the solution since $\rho = 0$ is not in the domain of interest [$a < \rho < b$] for the coaxial transmission line. The general solution for the longitudinal magnetic field function becomes

$$h_z(\rho, \phi) = [C J_n(k_c \rho) + D Y_n(k_c \rho)][A \sin n\phi + B \cos n\phi]$$

while the longitudinal magnetic field is

$$H_z(\rho, \phi, z) = [C J_n(k_c \rho) + D Y_n(k_c \rho)][A \sin n\phi + B \cos n\phi] e^{-j\beta z}$$

Since there is no longitudinal electric field for the TE modes, the only TE boundary conditions for the coaxial transmission line are

$$E_\phi(a, \phi, z) = 0 \quad E_\phi(b, \phi, z) = 0$$

where

$$\begin{aligned}
 E_{\phi} &= \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial \rho} \\
 &= \frac{j\omega\mu}{k_c} [CJ'_n(k_c \rho) + DY'_n(k_c \rho)] (A \sin n\phi + B \cos n\phi) e^{-j\beta z}
 \end{aligned}$$

The TE boundary conditions yield

$$CJ'_n(k_c a) + DY'_n(k_c a) = 0$$

$$CJ'_n(k_c b) + DY'_n(k_c b) = 0$$

This linear system of equations has a nontrivial solution only when the determinant is zero:

$$J'_n(k_c a)Y'_n(k_c b) - Y'_n(k_c a)J'_n(k_c b) = 0$$

The roots of this characteristic equation ($k_c = q'_{nm}$) define cutoff wavenumbers of the TE_{nm} modes for the coaxial line where m is the index on the roots and n is the order of the Bessel functions in the characteristic equation. Equivalently, we may write

$$\frac{C}{D} = -\frac{Y'_n(k_c a)}{J'_n(k_c a)} = -\frac{Y'_n(k_c b)}{J'_n(k_c b)}$$

which gives

$$\begin{aligned}
 R(\rho) &= CJ_n(k_c \rho) + DY_n(k_c \rho) = C \left[J_n(k_c \rho) + \frac{D}{C} Y_n(k_c \rho) \right] \\
 &= C \left[J_n(k_c \rho) - \frac{J'_n(k_c a)}{Y'_n(k_c a)} Y_n(k_c \rho) \right] \\
 &= C' \left[Y'_n(k_c a) J_n(k_c \rho) - J'_n(k_c a) Y_n(k_c \rho) \right]
 \end{aligned}$$

The resulting longitudinal magnetic field is

$$H_z(\rho, \phi, z) = [Y'_n(k_c a)J_n(k_c \rho) - J'_n(k_c a)Y_n(k_c \rho)][A \sin n\phi + B \cos n\phi] e^{-j\beta z}$$

where the constant C' has been incorporated into the constants A and B . The resulting transverse fields of the TE_{nm} modes are

$$E_\rho(\rho, \phi, z) = \frac{-j\omega\mu}{k_c^2 \rho} \frac{\partial H_z}{\partial \phi}$$

$$= \frac{-j\omega\mu n}{k_c^2 \rho} [Y'_n(k_c a)J_n(k_c \rho) - J'_n(k_c a)Y_n(k_c \rho)][A \cos n\phi - B \sin n\phi] e^{-j\beta z}$$

$$E_\phi(\rho, \phi, z) = \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial \rho}$$

$$= \frac{j\omega\mu}{k_c} [Y'_n(k_c a)J'_n(k_c \rho) - J'_n(k_c a)Y'_n(k_c \rho)][A \sin n\phi + B \cos n\phi] e^{-j\beta z}$$

$$H_\rho(\rho, \phi, z) = -\frac{j\beta}{k_c^2} \frac{\partial H_z}{\partial \rho}$$

$$= -\frac{j\beta}{k_c} [Y'_n(k_c a)J'_n(k_c \rho) - J'_n(k_c a)Y'_n(k_c \rho)][A \sin n\phi + B \cos n\phi] e^{-j\beta z}$$

$$H_\phi(\rho, \phi, z) = -\frac{j\beta}{k_c^2 \rho} \frac{\partial H_z}{\partial \phi}$$

$$= -\frac{j\beta n}{k_c^2 \rho} [Y'_n(k_c a)J_n(k_c \rho) - J'_n(k_c a)Y_n(k_c \rho)][A \cos n\phi - B \sin n\phi] e^{-j\beta z}$$

The cutoff frequency of the TE_{nm} mode may be defined in terms of the roots to the characteristic equation.

$$k_{c_{nm}} = 2\pi f_{c_{nm}} \sqrt{\mu\epsilon} = q'_{nm}$$

$$f_{c_{nm}} = \frac{q'_{nm}}{2\pi\sqrt{\mu\epsilon}}$$

The wavenumber for the TE_{nm} mode is

$$\beta_{nm} = \sqrt{k^2 - k_{c_{nm}}^2} = \sqrt{k^2 - q_{nm}'^2} = k \sqrt{1 - \left(\frac{f_{c_{nm}}}{f} \right)^2}$$

The wave impedance of the TE_{nm} mode is

$$Z_{TE_{nm}} = \frac{E_\rho}{H_\phi} = \frac{k\eta}{\beta} = \frac{\eta}{\sqrt{1 - \left(\frac{f_{c_{nm}}}{f} \right)^2}}$$

Coaxial Line TM modes

The longitudinal electric field of the TM modes within the coaxial transmission line must satisfy

$$\nabla^2 E_z + k^2 E_z = 0$$

where

$$E_z(\rho, \phi, z) = e_z(\rho, \phi) e^{-j\beta z}$$

The longitudinal electric field function of the coaxial line TM modes satisfies the same differential equation as the magnetic field of the TE modes. Thus, we may write the solutions for $e_z(\rho, \varphi)$ and $E_z(\rho, \varphi, z)$ as

$$e_z(\rho, \varphi) = R(\rho)P(\varphi)$$

$$= [CJ_n(k_c \rho) + DY_n(k_c \rho)][A \sin n\varphi + B \cos n\varphi]$$

$$E_z(\rho, \varphi, z) = [CJ_n(k_c \rho) + DY_n(k_c \rho)][A \sin n\varphi + B \cos n\varphi] e^{-j\beta z}$$

The TM boundary conditions for the coaxial line are

$$E_\varphi(a, \varphi, z) = 0 \quad E_\varphi(b, \varphi, z) = 0$$

$$E_z(a, \varphi, z) = 0 \quad E_z(b, \varphi, z) = 0$$

Again, we find that enforcement of the boundary condition on E_z automatically satisfies transverse field boundary condition. Application of the boundary condition on E_z yields

$$CJ_n(k_c a) + DY_n(k_c a) = 0$$

$$CJ_n(k_c b) + DY_n(k_c b) = 0$$

This linear system of equations has a nontrivial solution only when the determinant is zero:

$$J_n(k_c a)Y_n(k_c b) - Y_n(k_c a)J_n(k_c b) = 0$$

The roots of this characteristic equation ($k_c = q_{nm}$) define cutoff wavenumbers of the TM_{nm} modes for the coaxial line. Equivalently, we may write

$$\frac{C}{D} = -\frac{Y_n(k_c a)}{J_n(k_c a)} = -\frac{Y_n(k_c b)}{J_n(k_c b)}$$

which gives

$$\begin{aligned}
R(\rho) &= C J_n(k_c \rho) + D Y_n(k_c \rho) = C \left[J_n(k_c \rho) + \frac{D}{C} Y_n(k_c \rho) \right] \\
&= C \left[J_n(k_c \rho) - \frac{J_n(k_c a)}{Y_n(k_c a)} Y_n(k_c \rho) \right] \\
&= C' \left[Y_n(k_c a) J_n(k_c \rho) - J_n(k_c a) Y_n(k_c \rho) \right]
\end{aligned}$$

The resulting longitudinal electric field is

$$\begin{aligned}
E_z(\rho, \phi, z) \\
&= [Y_n(k_c a) J_n(k_c \rho) - J_n(k_c a) Y_n(k_c \rho)] [A \sin n\phi + B \cos n\phi] e^{-j\beta z}
\end{aligned}$$

and the transverse fields are

$$\begin{aligned}
E_\rho(\rho, \phi, z) &= \frac{-j\beta}{k_c^2} \frac{\partial E_z}{\partial \rho} \\
&= \frac{-j\beta}{k_c^2} [Y_n(k_c a) J_n'(k_c \rho) - J_n(k_c a) Y_n'(k_c \rho)] [A \sin n\phi + B \cos n\phi] e^{-j\beta z} \\
E_\phi(\rho, \phi, z) &= \frac{-j\beta}{k_c^2 \rho} \frac{\partial E_z}{\partial \phi} \\
&= \frac{-j\beta n}{k_c^2 \rho} [Y_n(k_c a) J_n(k_c \rho) - J_n(k_c a) Y_n(k_c \rho)] [A \cos n\phi - B \sin n\phi] e^{-j\beta z} \\
H_\rho(\rho, \phi, z) &= \frac{j\omega \epsilon}{k_c^2 \rho} \frac{\partial E_z}{\partial \phi} \\
&= \frac{j\omega \epsilon n}{k_c^2 \rho} [Y_n(k_c a) J_n(k_c \rho) - J_n(k_c a) Y_n(k_c \rho)] [A \cos n\phi - B \sin n\phi] e^{-j\beta z}
\end{aligned}$$

$$\begin{aligned}
H_{\phi}(\rho, \phi, z) &= \frac{-j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial \rho} \\
&= \frac{-j\omega\epsilon}{k_c} [Y_n(k_c a) J_n'(k_c \rho) - J_n(k_c a) Y_n'(k_c \rho)] [A \sin n\phi + B \cos n\phi] e^{-j\beta z}
\end{aligned}$$

The cutoff frequency of the TM_{nm} mode is given by

$$k_{c_{nm}} = 2\pi f_{c_{nm}} \sqrt{\mu\epsilon} = q_{nm}$$

$$f_{c_{nm}} = \frac{q_{nm}}{2\pi\sqrt{\mu\epsilon}}$$

The wavenumber for the TM_{nm} mode is

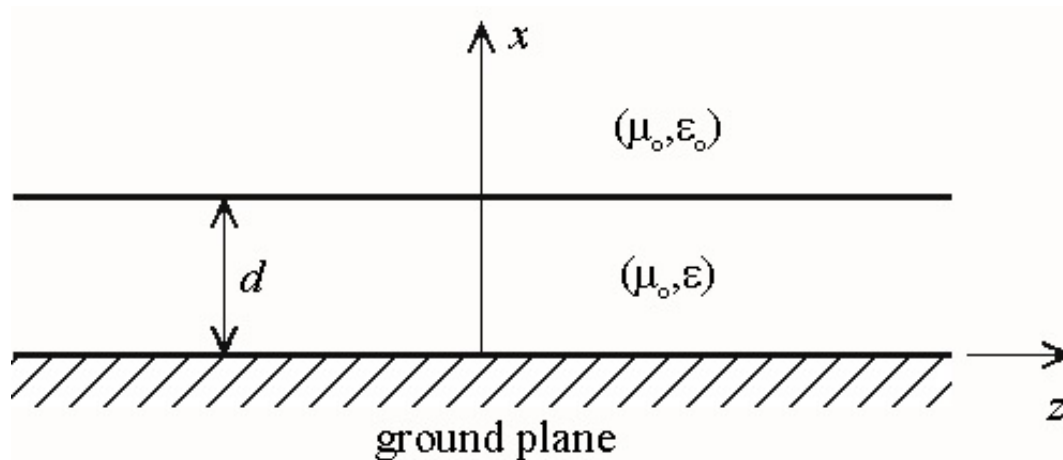
$$\beta_{nm} = \sqrt{k^2 - k_{c_{nm}}^2} = \sqrt{k^2 - p_{nm}^2} = k \sqrt{1 - \left(\frac{f_{c_{nm}}}{f}\right)^2}$$

The wave impedance of the TM_{nm} mode is

$$Z_{TM_{nm}} = \frac{E_{\rho}}{H_{\phi}} = \frac{\eta\beta}{k} = \eta \sqrt{1 - \left(\frac{f_{c_{nm}}}{f}\right)^2}$$

Grounded Dielectric Slab Waveguide

The grounded dielectric slab waveguide is the basis for many planar transmission lines (microstrip, stripline, etc.). This waveguide supports the propagation of *surface waves* which are guided along the dielectric interface. The grounded dielectric slab can support TE and TM modes but not the TEM mode since there is only one conductor. Actually, the conductor is not necessary for the TE and TM modes to propagate. A simple dielectric slab without a ground plane can also support TE and TM modes.



Assumptions:

- i. The grounded dielectric slab is of infinite extent in the y and z directions.
- ii. The nonmagnetic dielectric and the surrounding air are lossless.
- iii. Waves propagate in the z direction ($e^{-j\beta z}$).
- iv. Fields decay exponentially away from the slab (the fields of the propagating wave are concentrated within the dielectric).

The analysis of the grounded dielectric slab is somewhat different than that of the rectangular and circular waveguides and coaxial transmission line given the inhomogeneous dielectric through which the wave propagates.

Grounded Dielectric Slab TM modes

The TM longitudinal electric field associated with any uniform guiding structure carrying a wave in the z direction may be written as

$$E_z(x,y,z) = e_z(x,y) e^{-j\beta z}$$

where the electric field function must satisfy

$$\frac{\partial^2 e_z(x,y)}{\partial x^2} + \frac{\partial^2 e_z(x,y)}{\partial y^2} + k_c^2 e_z(x,y) = 0$$

with $k_c^2 = k^2 - \beta^2$. Given the symmetry of the dielectric slab, there should be no variation in the fields of the propagating wave with respect to y . The governing differential equation for the longitudinal electric field function of the wave becomes

$$\frac{\partial^2 e_z(x,y)}{\partial x^2} + k_c^2 e_z(x,y) = 0$$

With no y -variation in the fields, the only non-zero transverse components in the TM waves are E_x and H_y .

$$\begin{aligned} E_x &= \frac{-j\beta}{k_c^2} \frac{\partial E_z}{\partial x} & H_x &= \frac{j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial y} = 0 \\ E_y &= \frac{-j\beta}{k_c^2} \frac{\partial E_z}{\partial y} = 0 & H_y &= \frac{-j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial x} \end{aligned}$$

Given the inhomogeneous dielectric, the equation above must be applied to the individual homogeneous dielectric and air regions which are characterized by different wavenumbers.

$$k = k_o = \omega \sqrt{\mu_o \epsilon_o} \quad (\text{in the air})$$

$$k = k_d = \omega \sqrt{\mu_o \epsilon_o \epsilon_r} = \sqrt{\epsilon_r} k_o \quad (\text{in the dielectric})$$

This also means that the cutoff wavenumbers k_c in the two regions are different. If we assume propagation in the dielectric and attenuation in the air, then

$$k_o^2 - \beta^2 = -h^2 \quad (\text{in the air})$$

$$\epsilon_r k_o^2 - \beta^2 = k_c^2 \quad (\text{in the dielectric})$$

Given the signs on the cutoff wavenumbers, the governing equations in the air and dielectric regions are

$$\frac{\partial^2 e_z(x,y)}{\partial x^2} + k_c^2 e_z(x,y) = 0 \quad (0 \leq x \leq d)$$

$$\frac{\partial^2 e_z(x,y)}{\partial x^2} - h^2 e_z(x,y) = 0 \quad (x \geq d)$$

The general solutions to these equations are

$$e_z(x,y) = A_1 \sin k_c x + B_1 \cos k_c x \quad (0 \leq x \leq d)$$

$$e_z(x,y) = A_2 e^{hx} + B_2 e^{-hx} \quad (x \geq d)$$

The boundary conditions for the grounded dielectric slab are

$$(1) \quad E_z(0,y,z) = 0 \quad B_1 = 0$$

$$(2) \quad E_z(x,y,z) \rightarrow 0 \quad \text{as } x \rightarrow \infty \quad A_2 = 0$$

$$(3) \quad E_z(d_-,y,z) = E_z(d_+,y,z)$$

$$(4) \quad H_y(d_-,y,z) = H_y(d_+,y,z)$$

Boundary condition (1) ensures that the tangential electric field on the surface of the ground plane is zero. Boundary condition (2) ensures that the fields in the air region decay to zero as one gets very far away from the dielectric slab. Boundary conditions (3) and (4) enforce the continuity of the tangential electric and magnetic fields across the air-dielectric interface. Enforcement of the first two boundary conditions yields

$$E_z(x,y,z) = A_1 \sin k_c x e^{-j\beta z} \quad (0 \leq x \leq d)$$

$$E_z(x,y,z) = B_2 e^{-hx} e^{-j\beta z} \quad (x \geq d)$$

Enforcement of the remaining two boundary conditions gives the following results.

$$B.C. (3) \quad \Rightarrow \quad A_1 \sin k_c d = B_2 e^{-hd}$$

$$H_y = \frac{-j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial x} = \frac{-j\omega\epsilon_o\epsilon_r}{k_c} A_1 \cos k_c x e^{-j\beta z} \quad (0 \leq x \leq d)$$

$$H_y = \frac{j\omega\epsilon}{h^2} \frac{\partial E_z}{\partial x} = \frac{-j\omega\epsilon_o}{h} B_2 e^{-hx} e^{-j\beta z} \quad (x \geq d)$$

$$B.C. (4) \quad \Rightarrow \quad \frac{A_1 \epsilon_r}{k_c} \cos k_c d = \frac{B_2}{h} e^{-hd}$$

Dividing the equation resulting from the enforcement of B.C. (3) by the equation from B.C. (4) gives

$$\frac{k_c}{\epsilon_r} \tan k_c d = h \quad (1)$$

At this point we have two unknowns (k_c and h) but only one equation. We may write a second equation in terms of k_c and h by solving for β in the cutoff wavenumber equations. This yields

$$k_o^2 - \beta^2 = -h^2 \quad \Rightarrow \quad \beta = k_o^2 + h^2$$

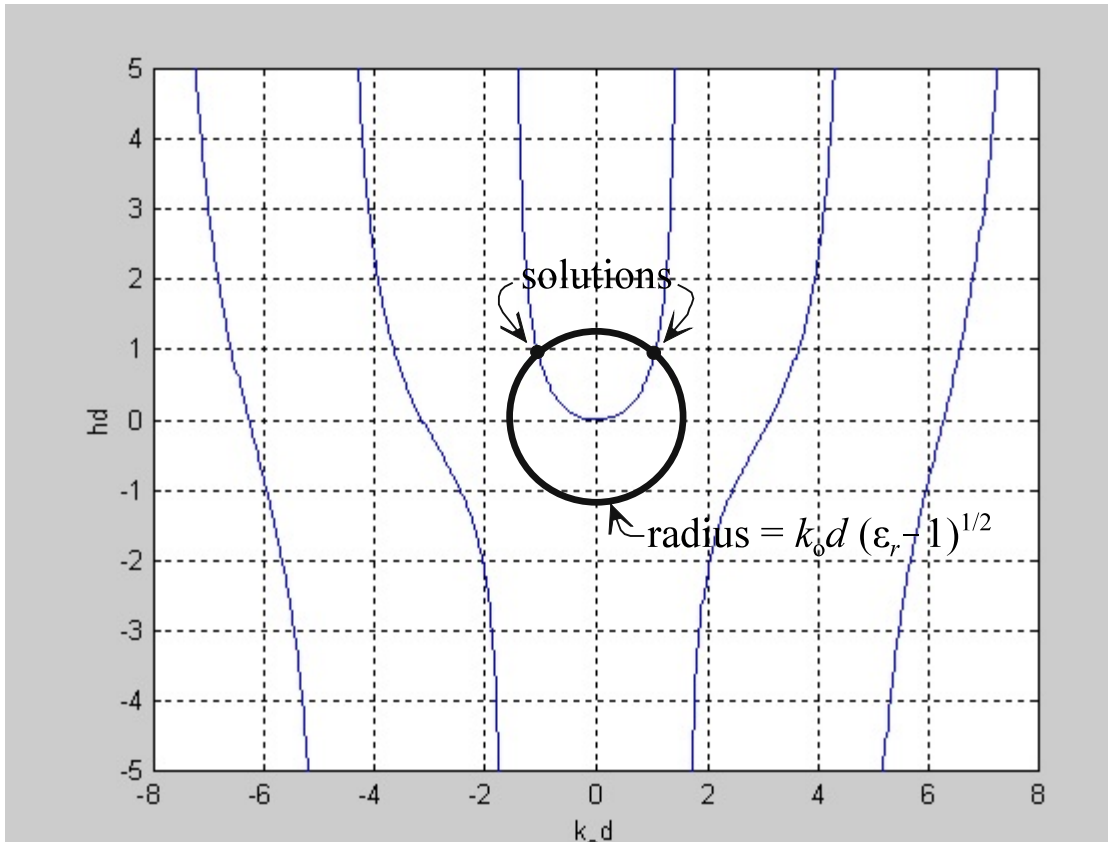
$$\epsilon_r k_o^2 - \beta^2 = k_c^2 \quad \Rightarrow \quad \beta = \epsilon_r k_o^2 - k_c^2$$

$$h^2 + k_c^2 = (\epsilon_r - 1)k_o^2 \quad (2)$$

If we multiply equation (1) by d and equation (2) by d^2 , we obtain two equations that can be solved graphically on a plot of hd verses $k_c d$.

$$k_c d \tan k_c d = \epsilon_r h d$$

$$(hd)^2 + (k_c d)^2 = (\epsilon_r - 1)(k_o d)^2$$



For valid solutions, h must be positive. The two intersections represent one mode. As the electrical thickness of the slab grows, more TM_n modes are possible. The TM_0 mode is the dominant mode with a cutoff frequency of

zero. The TM_n mode begins to propagate when the radius of the circle equals $n\pi$. Therefore, the cutoff frequency for the TM_n mode may be written as

$$\left[2\pi f_{c_n} \sqrt{\mu_o \epsilon_o} \right] d \sqrt{\epsilon_r - 1} = n\pi$$

$$f_{c_n} = \frac{nc}{2d\sqrt{\epsilon_r - 1}}$$

The resulting fields for the TM_n mode of the grounded dielectric slab waveguide may be found by writing the constant B_2 in terms of A_1 as given by boundary condition (3).

$$B_2 = A_1 e^{hd} \sin k_c d$$

$$E_z(x, y, z) = A_1 \sin(k_c x) e^{-j\beta z} \quad (0 \leq x \leq d)$$

$$E_z(x, y, z) = A_1 \sin(k_c d) e^{-h(x-d)} e^{-j\beta z} \quad (x \geq d)$$

$$E_x(x, y, z) = \frac{-j\beta}{k_c} A_1 \cos(k_c x) e^{-j\beta z} \quad (0 \leq x \leq d)$$

$$E_x(x, y, z) = \frac{-j\beta}{h} A_1 \sin(k_c d) e^{-h(x-d)} e^{-j\beta z} \quad (x \geq d)$$

$$H_y(x, y, z) = \frac{-j\omega \epsilon_o \epsilon_r}{k_c} A_1 \cos(k_c x) e^{-j\beta z} \quad (0 \leq x \leq d)$$

$$H_y(x, y, z) = \frac{-j\omega \epsilon_o}{h} A_1 \sin(k_c d) e^{-h(x-d)} e^{-j\beta z} \quad (x \geq d)$$

Grounded Dielectric Slab TE modes

The TE longitudinal magnetic field associated with any uniform guiding structure carrying a wave in the z direction may be written as

$$H_z(x,y,z) = h_z(x,y) e^{-j\beta z}$$

where the magnetic field function must satisfy

$$\frac{\partial^2 h_z(x,y)}{\partial x^2} + k_c^2 h_z(x,y) = 0$$

given no y -variation in the fields. With no y -variation in the fields, the only non-zero transverse components in the TE waves are E_y and H_x .

$$\begin{aligned} E_x &= \frac{-j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial y} = 0 & H_x &= -\frac{j\beta}{k_c^2} \frac{\partial H_z}{\partial x} \\ E_y &= \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial x} & H_y &= -\frac{j\beta}{k_c^2} \frac{\partial H_z}{\partial y} = 0 \end{aligned}$$

The governing equations for the longitudinal magnetic field in the air and dielectric regions are

$$\begin{aligned} \frac{\partial^2 h_z(x,y)}{\partial x^2} + k_c^2 h_z(x,y) &= 0 & (0 \leq x \leq d) \\ \frac{\partial^2 h_z(x,y)}{\partial x^2} - h^2 h_z(x,y) &= 0 & (x \geq d) \end{aligned}$$

where the wavenumber in the air region defines a propagating wave while the wavenumber in the dielectric defines an attenuated wave.

The general solutions to these equations are

$$h_z(x,y) = A_1 \sin k_c x + B_1 \cos k_c x \quad (0 \leq x \leq d)$$

$$h_z(x,y) = A_2 e^{hx} + B_2 e^{-hx} \quad (x \geq d)$$

The TE boundary conditions for the grounded dielectric slab are

$$(1) \quad E_y(0,y,z) = 0$$

$$(2) \quad E_y(x,y,z) \rightarrow 0 \quad \text{as } x \rightarrow \infty$$

$$(3) \quad E_y(d_-,y,z) = E_y(d_+,y,z)$$

$$(4) \quad H_z(d_-,y,z) = H_z(d_+,y,z)$$

where

$$E_y = \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial x} = \frac{j\omega\mu}{k_c} (A_1 \cos k_c x - B_1 \sin k_c x) e^{-j\beta z} \quad (0 \leq x \leq d)$$

$$E_y = \frac{j\omega\mu}{-h^2} \frac{\partial H_z}{\partial x} = -\frac{j\omega\mu}{h} (A_2 e^{hx} - B_2 e^{-hx}) e^{-j\beta z} \quad (z \geq d)$$

Enforcement of boundary condition (1) yields $A_1 = 0$ while the enforcement of boundary condition (2) yields $A_2 = 0$. Enforcement of the remaining two boundary conditions gives the following results.

$$B.C. (3) \quad \Rightarrow \quad -\frac{B_1}{k_c} \sin k_c d = \frac{B_2}{h} e^{-hd}$$

$$B.C. (4) \quad \Rightarrow \quad B_1 \cos k_c d = B_2 e^{-hd}$$

Dividing the equation resulting from the enforcement of B.C. (4) by the equation from B.C. (3) gives

$$-k_c \cot k_c d = h \quad (1)$$

Just as in the TM case, we may write a second equation in terms of k_c and h by solving for β in the cutoff wavenumber equations. This yields

$$k_o^2 - \beta^2 = -h^2 \quad \Rightarrow \quad \beta = k_o^2 + h^2$$

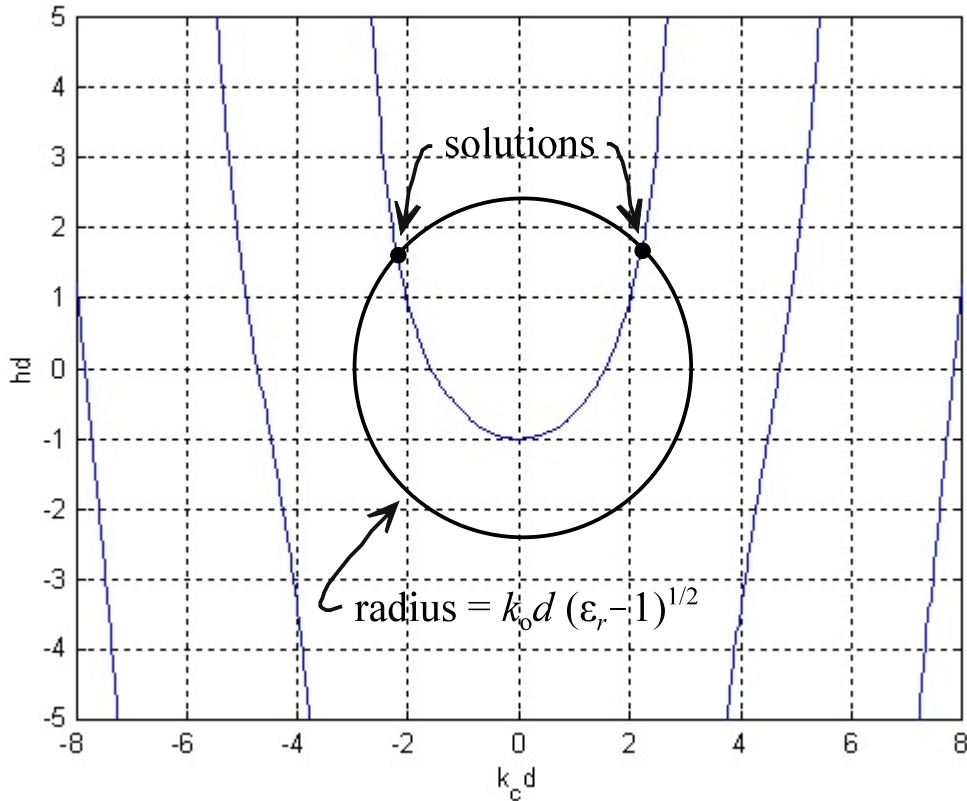
$$\epsilon_r k_o^2 - \beta^2 = k_c^2 \quad \Rightarrow \quad \beta = \epsilon_r k_o^2 - k_c^2$$

$$h^2 + k_c^2 = (\epsilon_r - 1)k_o^2 \quad (2)$$

If we multiply equation (1) by d and equation (2) by d^2 , we obtain two equations that can be solved graphically on a plot of hd versus $k_c d$.

$$-k_c d \cot k_c d = hd$$

$$(hd)^2 + (k_c d)^2 = (\epsilon_r - 1)(k_o d)^2$$



As the electrical thickness of the slab grows, the first TE mode (TE₁) mode propagates when the radius of the circle becomes greater than $\pi/2$. Note that there is no TE₀ mode (a mode with a cutoff frequency of zero) as in the case of the TM modes. The general TE_{*n*} mode propagates when the radius of the circle grows larger than $(2n-1)\pi/2$. Therefore, the cutoff frequency for the TE_{*n*} mode may be written as

$$\left[2\pi f_{c_n} \sqrt{\mu_o \epsilon_o} \right] d \sqrt{\epsilon_r - 1} = (2n - 1) \frac{\pi}{2}$$

$$f_{c_n} = \frac{(2n - 1)c}{4d \sqrt{\epsilon_r - 1}}$$

The resulting fields for the TE_{*n*} mode of the grounded dielectric slab waveguide may be found by writing the constant B_2 in terms of A_1 as given by boundary condition (4).

$$B_2 = B_1 e^{hd} \cos k_c d$$

$$H_z(x, y, z) = B_1 \cos(k_c x) e^{-j\beta z} \quad (0 \leq x \leq d)$$

$$H_z(x, y, z) = B_1 \cos(k_c d) e^{-h(x-d)} e^{-j\beta z} \quad (x \geq d)$$

$$H_x(x, y, z) = \frac{j\beta}{k_c} B_1 \sin(k_c x) e^{-j\beta z} \quad (0 \leq x \leq d)$$

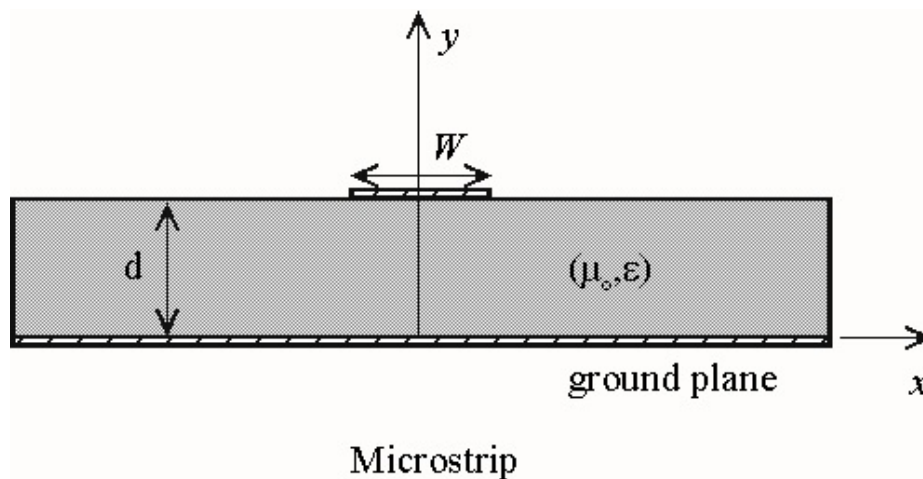
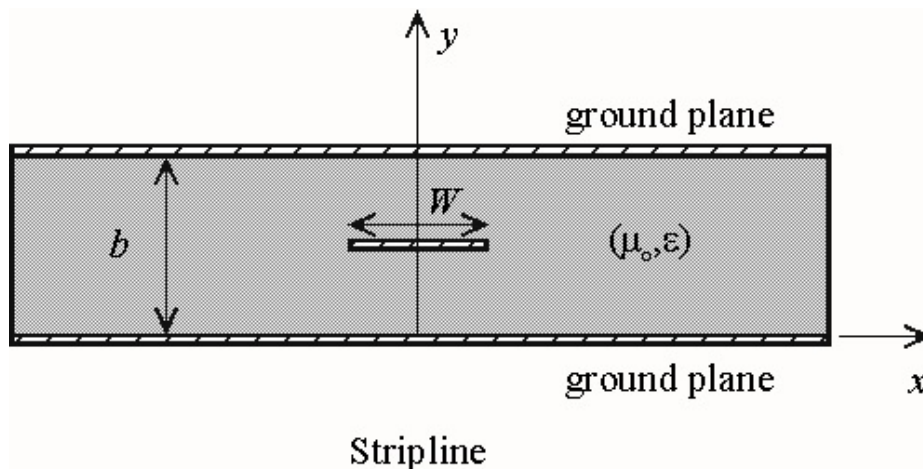
$$H_x(x, y, z) = \frac{-j\beta}{h} B_1 \cos(k_c d) e^{-h(x-d)} e^{-j\beta z} \quad (x \geq d)$$

$$E_y(x, y, z) = \frac{-j\omega\mu_o}{k_c} B_1 \sin(k_c x) e^{-j\beta z} \quad (0 \leq x \leq d)$$

$$E_y(x, y, z) = \frac{-j\omega\mu_o}{h} B_1 \cos(k_c d) e^{-h(x-d)} e^{-j\beta z} \quad (x \geq d)$$

Stripline and Microstrip

Stripline and microstrip are two commonly used wave guiding structures in low-power applications at microwave frequencies. The planar structure of these devices allows them to be fabricated using the same techniques in making printed circuits. The general structures of these devices are shown below.



The stripline configuration can be viewed as a flattened coaxial line. The stripline supports the same TEM mode as the coaxial line with electric field lines that emanate from the inner conductor to the ground planes (outer conductor) and magnetic field lines which encircle the inner

conductor. Even though the stripline may not be totally enclosed like a coaxial line, if the width of the ground plane is made large in comparison to the center conductor width W (about $5W$), then the fields at the outer edges of the stripline are small. Thus, stripline has the same advantage as coax in that there is little radiation from the line.

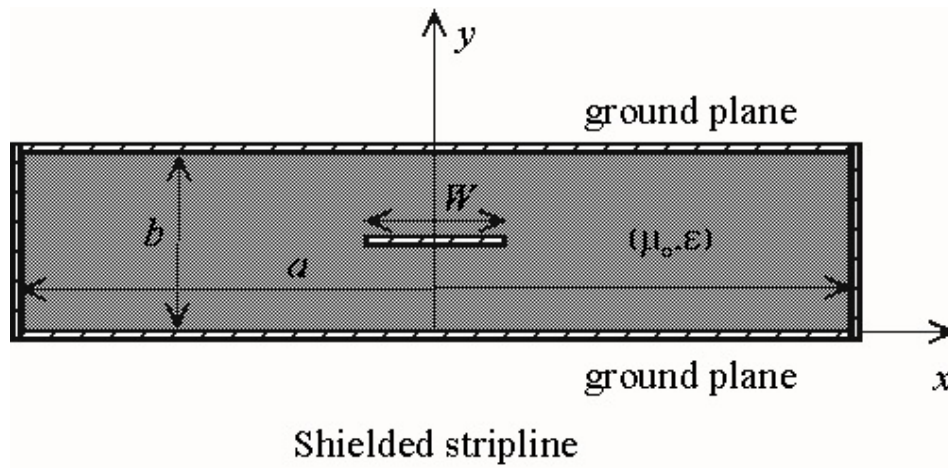
The microstrip configuration is unlike the stripline given that the transverse fields of the propagating wave are not confined to the homogeneous dielectric. A portion of the transverse fields lie in the air above the dielectric slab. This inhomogeneous dielectric prevents a pure TEM mode from propagating but a so-called *quasi-TEM* mode does propagate. This quasi-TEM mode is actually a hybrid mode which propagates at relatively low frequencies and has transverse fields which are essentially the same as the TEM mode that would propagate if the dielectric surrounding the upper conductor were uniform. In actuality, the stripline “TEM” mode is also a quasi-TEM mode (hybrid) but typically has a lower cutoff frequency than that of the microstrip.

Microstrip offers the advantage of an easy connection of devices to the line at any point given the exposed upper conductor. This is not the case for stripline since the inner conductor is enclosed. Stripline offers a higher bandwidth than an equivalent microstrip since the pure TEM mode has a cutoff frequency of zero.

Both stripline and microstrip are relatively low power structures because of the field distributions which result around the conductor configurations. The transverse fields of the propagating modes are concentrated around the stripline center conductor and the microstrip upper conductor where relatively large field magnitudes are encountered. The field levels that a given configuration can handle are limited by the breakdown voltage of the surrounding dielectric. Waveguides are capable of handling higher power levels because the fields are spread more evenly through the cross-section of the structure.

Stripline Characteristics

In order to simplify the analysis of the stripline, we consider a shielded stripline configuration as shown below. This configuration approximates that of the actual stripline configuration assuming that a is large in comparison to W . The analysis of this simplified structure is still quite involved. The shielded stripline configuration can support the pure TEM mode since it is totally enclosed. We can determine the characteristic impedance of this structure if an analytical expression for the per-unit-length capacitance can be found.



The characteristic impedance of the transmission line is given by

$$Z_o = \sqrt{\frac{L}{C}}$$

The phase velocity within the shielded stripline is found according to

$$v_p = \frac{1}{\sqrt{\mu_o \epsilon_o \epsilon_r}} = \frac{c}{\sqrt{\epsilon_r}} = \frac{1}{\sqrt{LC}}$$

Thus, we may write

$$Z_o = \sqrt{\frac{L}{C}} = \frac{\sqrt{LC}}{C} = \frac{1}{v_p C}$$

In the determination of the shielded stripline capacitance, we assume that the dielectric within the stripline is lossless and the thickness of the center conductor is zero. The resulting characteristic impedance is

$$Z_o = \frac{30\pi}{\sqrt{\epsilon_r}} \frac{K(k)}{K(k')}$$

where K is the elliptic integral of the first kind and

$$k = \operatorname{sech}\left(\frac{\pi W}{2b}\right) \quad k' = \tanh\left(\frac{\pi W}{2b}\right)$$

By applying a simple curve fit to the analytical results, we find

$$Z_o = \frac{30\pi}{\sqrt{\epsilon_r}} \frac{1}{\left(\frac{W_e}{b}\right) + 0.441}$$

where W_e is the effective width of the center conductor given by

$$\frac{W_e}{b} = \begin{cases} \frac{W}{b} & \frac{W}{b} > 0.35 \\ \frac{W}{b} - \left(0.35 - \frac{W}{b}\right)^2 & \frac{W}{b} < 0.35 \end{cases}$$

For stripline design, we normally know what characteristic impedance is required and thus need to know W/b in terms of Z_o rather than Z_o in terms of W/b . Thus, solving the equations above for W/b yields

$$\frac{W}{b} = \begin{cases} \frac{30\pi}{\sqrt{\epsilon_r} Z_o} - 0.441 & \sqrt{\epsilon_r} Z_o < 120 \\ 0.85 - \sqrt{1.041 - \frac{30\pi}{\sqrt{\epsilon_r} Z_o}} & \sqrt{\epsilon_r} Z_o > 120 \end{cases}$$

The attenuation constant in a stripline (with a center conductor of thickness t) due to conductor losses only may be written as

$$\alpha_c = \begin{cases} \frac{90 R_s \epsilon_r Z_o A}{\pi(b-t)} & \sqrt{\epsilon_r} Z_o < 120 \\ \frac{0.16 R_s B}{Z_o b} & \sqrt{\epsilon_r} Z_o > 120 \end{cases}$$

where

$$A = 1 + \frac{2W}{b-t} + \frac{1}{\pi} \frac{b+t}{b-t} \ln \left(\frac{2b-t}{t} \right)$$

$$B = 1 + \frac{b}{0.5W + 0.7t} \left(0.5 + \frac{0.414t}{W} + \frac{1}{2\pi} \ln \frac{4\pi W}{t} \right)$$

Microstrip Characteristics

The inhomogeneous dielectric of the microstrip geometry along with the fact that the quasi-static mode is a hybrid mode complicates the analysis of the microstrip geometry. However, if the mode is truly quasi-static, then we should be able to accurately approximate the phase velocity and phase constant of the propagating waves for a microstrip using the standard TEM equations with an effective dielectric constant ϵ_e .

$$v_p = \frac{1}{\sqrt{\mu_o \epsilon_o \epsilon_e}} = \frac{c}{\sqrt{\epsilon_e}}$$
$$\beta = k = \omega \sqrt{\mu_o \epsilon_o \epsilon_e} = k_o \sqrt{\epsilon_e}$$

where the effective dielectric constant depends on the microstrip substrate thickness d and the conductor width W and satisfies the relation

$$1 < \epsilon_e < \epsilon_r$$

The fact that the effective dielectric constant lies between that of air and the dielectric means that the effective dielectric constant depends on how much of the propagating mode fields lie within the dielectric and air regions. The effective dielectric constant is approximated by

$$\epsilon_e = \frac{\epsilon_r + 1}{2} + \frac{\epsilon_r - 1}{2\sqrt{1 + 12d/W}}$$

Note that a very wide conductor with a very thin substrate ($d \ll W$) approximates the parallel plate waveguide yielding an effective dielectric constant of

$$\epsilon_e \approx \epsilon_r$$

The characteristic impedance of the microstrip may be written as

$$Z_o = \begin{cases} \frac{60}{\sqrt{\epsilon_e}} \ln \left(\frac{8d}{W} + \frac{W}{4d} \right) & W/d \leq 1 \\ \frac{120\pi}{\sqrt{\epsilon_e} [W/d + 1.393 + 0.667 \ln(W/d + 1.444)]} & W/d \geq 1 \end{cases}$$

Solving this equation for W/d yields

$$\frac{W}{d} = \begin{cases} \frac{8e^A}{e^{2A} - 2} & W/d \leq 2 \\ \frac{2}{\pi} \left[B - 1 - \ln(2B - 1) + \frac{\epsilon_r - 1}{2\epsilon_r} \left\{ \ln(B - 1) + 0.39 - \frac{0.61}{\epsilon_r} \right\} \right] & W/d \geq 2 \end{cases}$$

where

$$A = \frac{Z_o}{60} \sqrt{\frac{\epsilon_r + 1}{2}} + \frac{\epsilon_r - 1}{\epsilon_r + 1} \left(0.23 + \frac{0.11}{\epsilon_r} \right)$$

$$B = \frac{377\pi}{2Z_o\sqrt{\epsilon_r}}$$

The attenuation constant due to conductor loss in the microstrip geometry is

$$\alpha_c = \frac{R_s}{Z_o W}$$

where R_s is the surface resistivity of the conductor.

Dispersion and Group Velocity

In general, TE and TM waves of different frequencies propagate at different velocities and are attenuated at different rates on a wave guiding structure (*dispersion*). For example, the phase velocities of both TE and TM modes in a rectangular waveguide are

$$v_p = \frac{\omega}{\beta_{mn}} = \frac{\omega}{k \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}} = \frac{1/\sqrt{\mu\epsilon}}{\sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

On the other hand, the phase velocity of TEM waves (such as those on a lossless or low-loss coaxial transmission line) are independent of frequency:

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}}$$

Thus, there is typically no significant dispersion for low-loss guiding structures of reasonable length which utilize TEM waves. Dispersion is a concern when multi-frequency or broadband signals are propagated using a TE or TM mode. These types of signals suffer distortion as they propagate along the structure since different components of the signals propagate at different velocities.

As an example, an amplitude-modulation (AM) signal is the sum of signal (information) frequencies (ω_s) and a carrier frequency (ω_c). The envelope of the carrier signal is the signal information carried by the signal. The phase velocity of the carrier signal is given by

$$v_p = \frac{\omega_c}{\beta}$$

The envelope (information) propagates at what is called the *group velocity* (v_g) which is defined by

$$v_g = \left[\frac{d\beta}{d\omega} \right]^{-1}$$

Example

Determine the group velocity for a given mode in an air-filled rectangular waveguide.

$$\begin{aligned} \frac{d\beta}{d\omega} &= \frac{d}{d\omega} \left[k_o \sqrt{1 - \left(\frac{f_{c_{mn}}}{f} \right)^2} \right] = \frac{d}{d\omega} \left[k_o \sqrt{1 - \left(\frac{\omega_{c_{mn}}}{\omega} \right)^2} \right] \\ &= \frac{d}{d\omega} \left[\sqrt{\mu_o \epsilon_o} \sqrt{\omega^2 - \omega_{c_{mn}}^2} \right] \\ &= \sqrt{\mu_o \epsilon_o} \frac{1}{2} (\omega^2 - \omega_{c_{mn}}^2)^{-1/2} (2\omega) \\ &= \frac{\omega \sqrt{\mu_o \epsilon_o}}{\sqrt{\omega^2 - \omega_{c_{mn}}^2}} = \frac{1}{c \sqrt{1 - \left(\frac{f_{c_{mn}}}{f} \right)^2}} \\ v_g &= \left[\frac{d\beta}{d\omega} \right]^{-1} = c \sqrt{1 - \left(\frac{f_{c_{mn}}}{f} \right)^2} \end{aligned}$$

Note that while the phase velocity within the air-filled waveguide can be greater than the speed of light, the information (envelope) travels at the group velocity which is less than the speed of light.