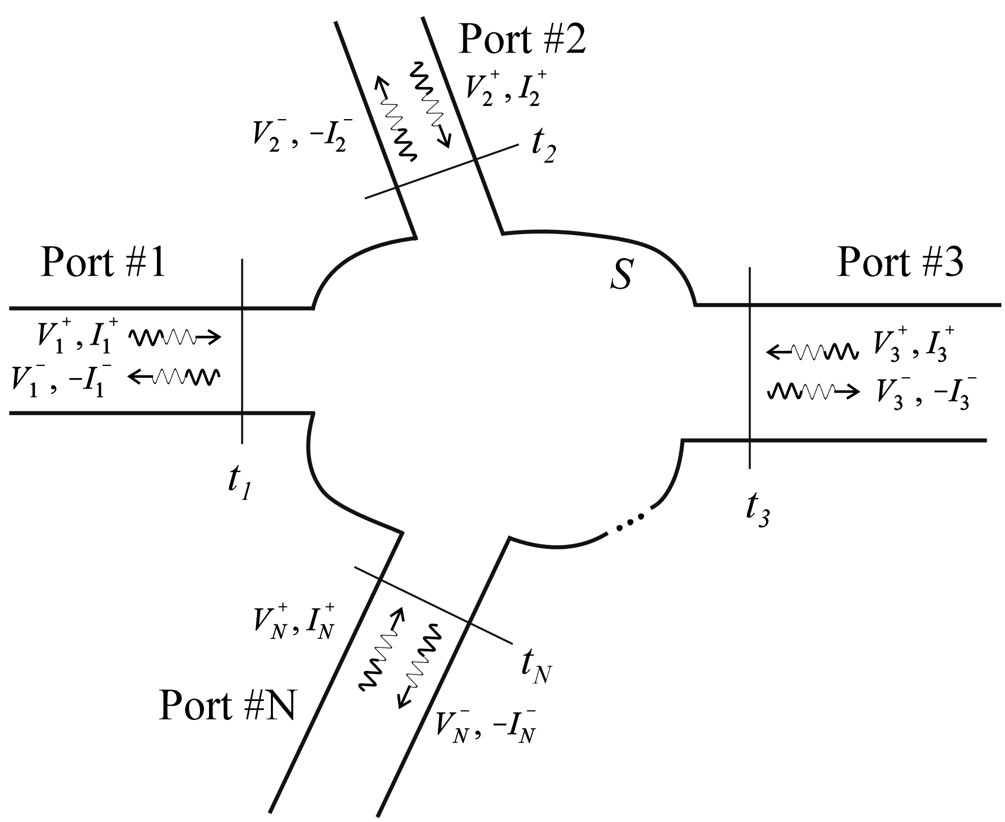
Microwave Network Analysis

N-Port Microwave Network

The general N-port microwave network is shown below where N is the total number of ports. The ports may be fed by any combination of transmission lines or waveguides. We assume that each waveguiding structure carries only the single dominant mode. A *terminal plane* (transverse plane) is defined for each port where the equivalent voltage and current will be defined. The terminal plane is designated as t_n for the n^{th} port. Discontinuities in the guiding structure will generally generate evanescent modes. If we choose the terminal planes far enough away from these discontinuities, then the evanescent modes decay sufficiently to be neglected.



If the coordinates of the wave guiding structures are chosen such that the terminal planes are each located at $z = 0$, then the voltage and current at the n^{th} terminal plane may be written as

$$V_n = V^+ + V^-$$

$$I_n = I^+ - I^-$$

Note that the reverse wave traveling out of the n^{th} port is dependent on the the reflection from the n^{th} port and waves that are coupled into the n^{th} port from the other ports. Thus, the impedance of the overall N-port network must be defined by an impedance matrix [Z] such that

$$[V] = [Z][I]$$

or

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & \cdots & Z_{1N} \\ Z_{21} & Z_{22} & Z_{23} & \cdots & Z_{2N} \\ Z_{31} & Z_{32} & Z_{33} & \cdots & Z_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_{N1} & Z_{N2} & Z_{N3} & \cdots & Z_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_N \end{bmatrix}$$

The individual elements of the impedance matrix may be determined according to

$$Z_{ij} = \frac{V_i}{I_j} \quad \text{with} \quad I_k = 0 \text{ for } k \neq j$$

In other words, we may drive port j with a current I_j while open-circuiting all other ports except j and measure the resulting open-circuit response at port i. The ratio of the open-circuit voltage at port i to the current at port j gives us the impedance matrix element Z_{ij}.

We may also define an admittance matrix according to

$$[I] = [Y][V]$$

or

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & \cdots & Y_{1N} \\ Y_{21} & Y_{22} & Y_{23} & \cdots & Y_{2N} \\ Y_{31} & Y_{32} & Y_{33} & \cdots & Y_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_{N1} & Y_{N2} & Y_{N3} & \cdots & Y_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{bmatrix}$$

The individual elements of the admittance matrix may be determined according to

$$Y_{ij} = \frac{I_i}{V_j} \quad \text{with} \quad V_k = 0 \text{ for } k \neq j$$

Thus, we may drive port j with a voltage V_j while short-circuiting all other ports except j and measure the resulting short-circuit current response at port i. The ratio of the short-circuit current at port i to the voltage at port j gives us the admittance matrix element Y_{ij}.

According to the definition of the impedance and admittance matrices, these matrices are inverses so that

$$[Y] = [Z]^{-1}$$

If the N-port microwave network is passive (no sources) and contains only isotropic media, the network is a *reciprocal network* and both the impedance and admittance matrices are symmetric. Examples of anisotropic materials (parameters are functions of direction - tensor μ and/or ε) are ferrites and plasmas. If the network is lossless, then the impedance and admittance matrices are purely imaginary.

Scattering Matrix

The equivalent currents and voltages used to define the impedance and admittance matrices for the general N-port network are somewhat abstract in that they cannot be easily measured for a given network at microwave frequencies. However, we may easily measure the amplitude and phase angle of the wave reflected (or scattered) from a port relative to the amplitude and phase angle of the wave incident on that port. Thus, we define a *scattering matrix* which relates the scattered voltage coefficients (V^-) to the incident wave voltage coefficients (V^+) according to

$$[V^-] = [S][V^+]$$

or

$$\begin{bmatrix} V_1^- \\ V_2^- \\ V_3^- \\ \vdots \\ V_N^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & \cdots & S_{1N} \\ S_{21} & S_{22} & S_{23} & \cdots & S_{2N} \\ S_{31} & S_{32} & S_{33} & \cdots & S_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & S_{N3} & \cdots & S_{NN} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ V_3^+ \\ \vdots \\ V_N^+ \end{bmatrix}$$

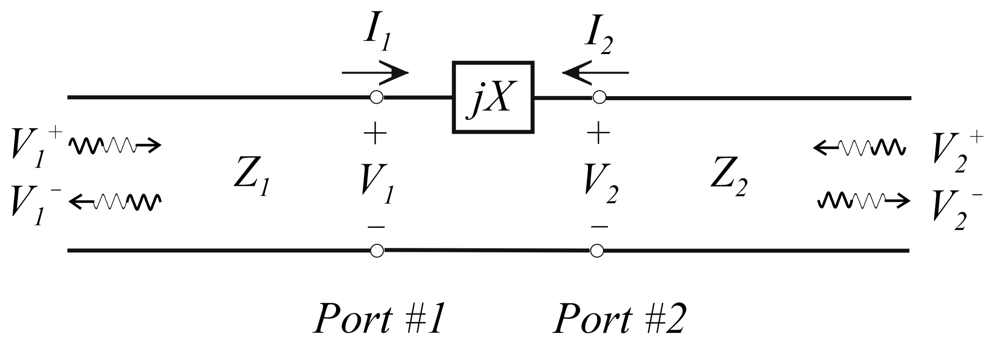
The individual elements of the scattering matrix may be determined according to

$$S_{ij} = \frac{V_i^-}{V_j^+} \qquad \text{with} \qquad V_k^+ = 0 \ \text{ for } \ k \neq j$$

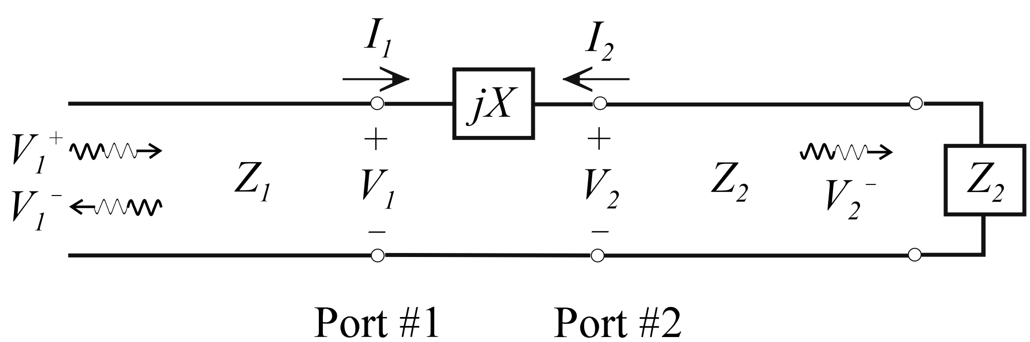
Thus, we may launch an incident wave toward port j while all other ports have no incident waves (the transmission lines or waveguides on these ports should be terminated by a matched load) and measure the scattered wave at port i. The ratio of the scattered wave at port i to the incident wave at port j gives us the scattering matrix element S_{ij}. The elements of the scattering matrix are referred to as the *s-parameters* of the network.

Example (Determination of s-parameters)

Determine the s-parameters for the 2-port network characterized by the series connection of transmission lines and a lumped reactance shown below.



Determination of S_{11}, S_{21} (excite port #1, matched termination on port #2)



Note that the matched termination eliminates any "incident" wave on port #2 ($V_2^+ = 0$).

$$S_{11} = \frac{V_1^-}{V_1^+}\bigg|_{V_2^+=0} = \Gamma_1\Big|_{V_2^+=0} = \frac{Z_{in,1} - Z_1}{Z_{in,1} + Z_1} = \frac{(Z_2 + jX) - Z_1}{(Z_2 + jX) + Z_1}$$

$$= \frac{Z_2 - Z_1 + jX}{Z_1 + Z_2 + jX}$$

$$S_{21} = \frac{V_2^-}{V_1^+}\bigg|_{V_2^+ = 0}$$

For the series reactance connection, we may relate the two port currents by

$$I_1 = -I_2$$

If we assume that both ports are located at a coordinate reference of $z = 0$ for each transmission line, then the current relation can be written as

$$I_1 = \frac{1}{Z_1}\left[V_1^+ - V_1^-\right] = -I_2 = -\frac{1}{Z_2}\left[-V_2^-\right]$$

$$\frac{V_1^+}{Z_1}\left[1 - \frac{V_1^-}{V_1^+}\right] = \frac{V_1^+}{Z_1}\left[1 - S_{11}\right] = \frac{V_2^-}{Z_2}$$

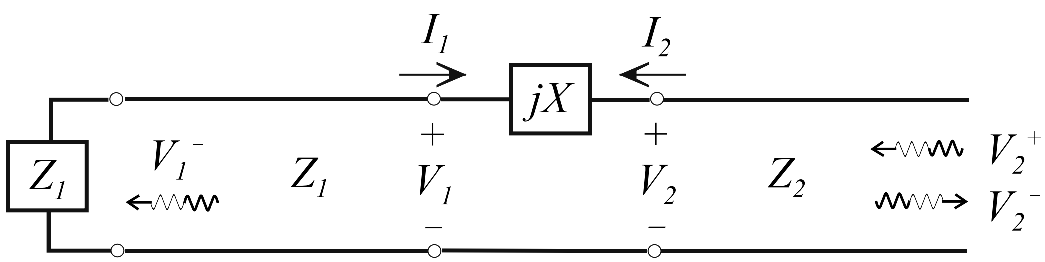
$$S_{21} = \frac{V_2^-}{V_1^+} = \frac{Z_2}{Z_1}\left[1 - S_{11}\right]$$

$$= \frac{Z_2}{Z_1}\left[1 - \frac{Z_2 - Z_1 + jX}{Z_1 + Z_2 + jX}\right]$$

$$= \frac{Z_2}{Z_1}\left[\frac{2Z_1}{Z_1 + Z_2 + jX}\right]$$

$$= \frac{2Z_2}{Z_1 + Z_2 + jX}$$

<u>Determination of S_{22}, S_{12}</u> (excite port #2, matched termination on port #1)



The matched termination eliminates any "incident" wave on port #1 ($V_1^+ = 0$).

$$S_{22} = \left.\frac{V_2^-}{V_2^+}\right|_{V_1^+=0} = \Gamma_2\Big|_{V_1^+=0} = \frac{Z_{in,2} - Z_2}{Z_{in,2} + Z_2} = \frac{(Z_1 + jX) - Z_2}{(Z_1 + jX) + Z_2}$$

$$= \frac{Z_1 - Z_2 + jX}{Z_1 + Z_2 + jX}$$

$$S_{12} = \left.\frac{V_1^-}{V_2^+}\right|_{V_1^+=0}$$

Again, we may relate the two port currents and find

$$I_1 = \frac{1}{Z_1}\left[-V_1^-\right] = -I_2 = -\frac{1}{Z_2}\left[V_2^+ - V_2^-\right]$$

$$\frac{V_1^-}{Z_1} = \frac{V_2^+}{Z_2}\left[1 - \frac{V_2^-}{V_2^+}\right] = \frac{V_2^+}{Z_2}\left[1 - S_{22}\right]$$

$$S_{12} = \frac{V_1^-}{V_2^+} = \frac{Z_1}{Z_2}\left[1 - S_{22}\right]$$

$$= \frac{Z_1}{Z_2}\left[1 - \frac{Z_1 - Z_2 + jX}{Z_1 + Z_2 + jX}\right]$$

$$= \frac{Z_1}{Z_2}\left[\frac{2Z_2}{Z_1 + Z_2 + jX}\right]$$

$$= \frac{2Z_1}{Z_1 + Z_2 + jX}$$

The overall scattering matrix for the two-port network is

$$[S] = \begin{bmatrix} \dfrac{Z_2 - Z_1 + jX}{Z_1 + Z_2 + jX} & \dfrac{2Z_1}{Z_1 + Z_2 + jX} \\ \dfrac{2Z_2}{Z_1 + Z_2 + jX} & \dfrac{Z_1 - Z_2 + jX}{Z_1 + Z_2 + jX} \end{bmatrix}$$

Properties of the Scattering Matrix

If we normalize all ports of the N-port microwave network to the same characteristic impedance, then

For a reciprocal network $\Rightarrow$ [S] is symmetric

For a lossless network $\Rightarrow$ [S] is unitary

The matrix [S] is unitary if it satisfies

$$[S]^t \, [S]^* = [U]$$

where

$[S]^t$ = the transpose of [S]

$[S]^*$ = the conjugate of [S]

[U] = identity matrix

For a unitary matrix [S], the product of any column of [S] with the conjugate of that column gives unity. The product of any column of [S] with the conjugate of any other column gives zero.

Scattering Matrix in Terms of the Impedance Matrix

If we assume that the characteristic impedances of all N-ports are identical and choose this characteristic impedance to be unity ($Z_{on} = 1$), then

$$V_n = V_n^+ + V_n^- \qquad \Rightarrow \qquad [V] = [V^+] + [V^-]$$

$$I_n = V_n^+ - V_n^- \qquad \Rightarrow \qquad [I] = [V^+] - [V^-]$$

When these equations for the voltage and current vectors are incorporated into the impedance matrix definition, we find

$$[V] = [Z][I]$$

$$[V^+] + [V^-] = [Z][V^+] - [Z][V^-]$$

Grouping the terms involving the forward voltage coefficients and reverse voltage coefficients yields,

$$([Z] + [U])[V^-] = ([Z] - [U])[V^+]$$

$$[V^-] = ([Z] + [U])^{-1}([Z] - [U])[V^+]$$

According to the definition of the scattering matrix,

$$[V^-] = [S][V^+]$$

so that the scattering matrix in terms of the impedance matrix is

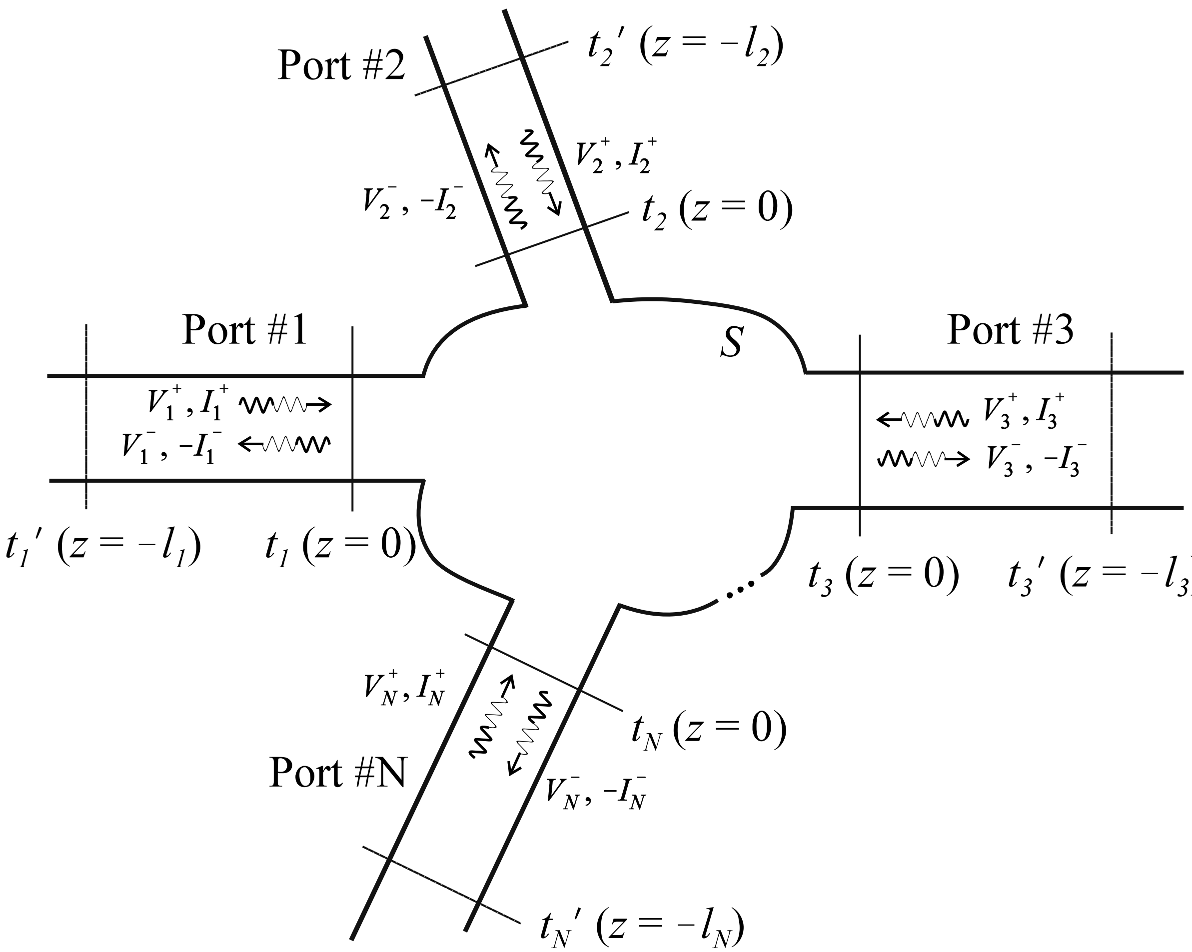
$$[S] = ([Z] + [U])^{-1}([Z] - [U])$$

We can also solve this equation for the impedance matrix in terms of the scattering matrix which yields

$$[Z] = ([U] - [S])^{-1}([U] + [S])$$

S-Parameters at Arbitrary Terminal Planes

We have assumed that all terminal planes for the wave guiding structures connected to the N-port microwave network are located at $z = 0$. If we wish to shift the terminal planes to some arbitrary locations at distances l_n away from the $z = 0$ reference, a new scattering matrix must be determined. Note that the terminal planes have been moved away from the N-port network. Shifting the planes closer to the N-port would require the opposite sign on the z-coordinates.



The incident and scattered voltage waves at the original and shifted terminal planes for the N ports are related by different scattering matrices. If we denote all quantities at the shifted terminal planes with a prime, then we may write

$$[V^-] = [S][V^+]$$

$$[V'^-] = [S'][V'^+]$$

According to the general equations for the equivalent voltage as a function of position on a wave guiding structure, the voltage on the n^{th} port is given by

$$V_n(z) = V_n^+ e^{-j\beta_n z} + V_n^- e^{j\beta_n z}$$

Thus, the coefficients of the incident and scattered voltage waves at the original terminal planes (unprimed terms) and the shifted terminal planes (primed terms) are related by

$$V_n^{+\prime} = V_n^+ e^{j\beta_n l_n} = V_n^+ e^{j\theta_n} \qquad \Rightarrow \qquad V_n^+ = e^{-j\theta_n} V_n^{+\prime}$$

$$V_n^{-\prime} = V_n^- e^{-j\beta_n l_n} = V_n^- e^{-j\theta_n} \qquad \Rightarrow \qquad V_n^- = e^{j\theta_n} V_n^{-\prime}$$

In matrix form, the coefficients of the incident and scattered voltage waves are related by

$$\begin{bmatrix} V_1^+ \\ V_2^+ \\ V_3^+ \\ \vdots \\ V_N^+ \end{bmatrix} = \begin{bmatrix} e^{-j\theta_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{-j\theta_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{-j\theta_N} \end{bmatrix} \begin{bmatrix} V_1^{+\prime} \\ V_2^{+\prime} \\ V_3^{+\prime} \\ \vdots \\ V_N^{+\prime} \end{bmatrix}$$

$$\begin{bmatrix} V_1^- \\ V_2^- \\ V_3^- \\ \vdots \\ V_N^- \end{bmatrix} = \begin{bmatrix} e^{j\theta_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{j\theta_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{j\theta_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{j\theta_N} \end{bmatrix} \begin{bmatrix} V_1^{-\prime} \\ V_2^{-\prime} \\ V_3^{-\prime} \\ \vdots \\ V_N^{-\prime} \end{bmatrix}$$

Inserting these incident and scattered wave vectors into the definition of the scattering matrix gives

$$\begin{bmatrix} e^{j\theta_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{j\theta_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{j\theta_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{j\theta_N} \end{bmatrix} \begin{bmatrix} V_1^{-\prime} \\ V_2^{-\prime} \\ V_3^{-\prime} \\ \vdots \\ V_N^{-\prime} \end{bmatrix} = [S] \begin{bmatrix} e^{-j\theta_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{-j\theta_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{-j\theta_N} \end{bmatrix} \begin{bmatrix} V_1^{+\prime} \\ V_2^{+\prime} \\ V_3^{+\prime} \\ \vdots \\ V_N^{+\prime} \end{bmatrix}$$

Solving the equation above for the scattered wave coefficients gives

$$\begin{bmatrix} V_1^{-\prime} \\ V_2^{-\prime} \\ V_3^{-\prime} \\ \vdots \\ V_N^{-\prime} \end{bmatrix} = \begin{bmatrix} e^{-j\theta_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{-j\theta_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{-j\theta_N} \end{bmatrix} [S] \begin{bmatrix} e^{-j\theta_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{-j\theta_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{-j\theta_N} \end{bmatrix} \begin{bmatrix} V_1^{+\prime} \\ V_2^{+\prime} \\ V_3^{+\prime} \\ \vdots \\ V_N^{+\prime} \end{bmatrix}$$

where

$$\begin{bmatrix} e^{j\theta_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{j\theta_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{j\theta_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{j\theta_N} \end{bmatrix}^{-1} = \begin{bmatrix} e^{-j\theta_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{-j\theta_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{-j\theta_N} \end{bmatrix}$$

The equation for the scattered wave coefficients defines the scattering matrix at the shifted terminal planes $[S']$ in terms of the scattering matrix at the original terminal planes $[S]$.

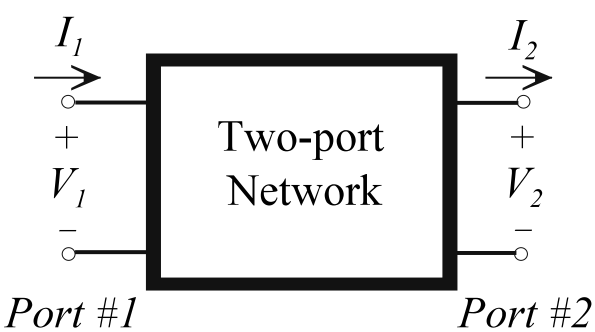
$$[S'] = \begin{bmatrix} e^{-j\theta_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{-j\theta_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{-j\theta_N} \end{bmatrix} [S] \begin{bmatrix} e^{-j\theta_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{-j\theta_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{-j\theta_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{-j\theta_N} \end{bmatrix}$$

This equation shows that there is a characteristic phase shift [defined by the electrical length associated with the physical length of the terminal plane shift, $(\theta_n = \beta_n l_n)$] for the incident and scattered waves. That is, the incident waves reach the shifted terminal plane before the original plane, and the scattered waves reach the shifted terminal plane after the original plane.

The Transmission Matrix (ABCD Matrix)

Many microwave network problems involve series connections (cascading) of several two-port networks. For this type of network, it is convenient to define a special set of two-port parameters known as the *transmission parameters.* The transmission parameters (defined by A,B,C and D) make up the *transmission matrix* which is also called the *ABCD matrix*. The transmission matrix of a network formed by several cascaded two-ports is simply the product of the transmission matrices of the individual two-ports.

The transmission matrix of a given two-port network relates the input voltage and current to the output voltage and current according to



$$V_1 = AV_2 + BI_2$$
$$I_1 = CV_2 + DI_2$$

Note that the convention for the direction of the output current (I_2) has been changed from our previous convention. This change in the current direction allows one to equate the output current of one stage to the input current of the following stage. In matrix form, the transmission equations are

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

where

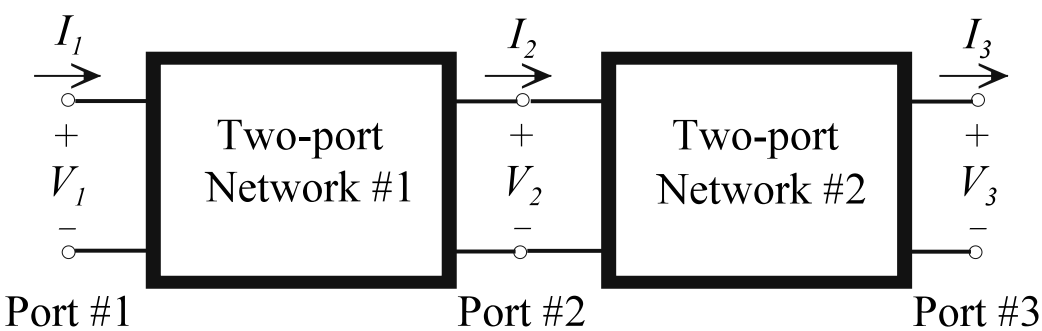
$$A = \left. \frac{V_1}{V_2} \right|_{I_2 = 0} \qquad \text{(open circuit voltage ratio)}$$

$$B = \left. \frac{V_1}{I_2} \right|_{V_2 = 0} \qquad \text{(short circuit transfer impedance)}$$

$$C = \left. \frac{I_1}{V_2} \right|_{I_2 = 0} \qquad \text{(open circuit transfer admittance)}$$

$$D = \left. \frac{I_1}{I_2} \right|_{V_2 = 0} \qquad \text{(short circuit current ratio)}$$

Given a pair of cascaded two-port networks as shown below, the transmission matrices for the network #1 and network #2 can be defined as



$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \qquad \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$

Simple substitution yields

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$

which defines the input quantities of the overall network to the output quantities. This technique is easily extended to the series connection of an arbitrary number of two-port networks.

Generalized Scattering Parameters

The incident and scattered waves in the definition of the N-port network scattering parameters may be normalized so that the power delivered to each port is independent of the characteristic impedances. Using the scattering matrix definition, the voltage and current at the terminal plane of the n^{th} port ($z = 0$) is

$$V_n = V_n^+ + V_n^-$$

$$I_n = \frac{1}{Z_{on}} (V_n^+ - V_n^-)$$

Assuming the characteristic impedances of the N ports are real, the power delivered to the n^{th} port is

$$P_n = \frac{1}{2} \mathrm{Re}\, [V_n I_n^*]$$

$$= \frac{1}{2Z_{on}} \mathrm{Re}\left[(V_n^+ + V_n^-)(V_n^{+*} - V_n^{-*}) \right]$$

$$= \frac{1}{2Z_{on}} \mathrm{Re}\left[|V_n^+|^2 - |V_n^-|^2 + (V_n^- V_n^{+*} - V_n^+ V_n^{-*}) \right]$$

$$= \frac{1}{2Z_{on}} \mathrm{Re}\left[|V_n^+|^2 - |V_n^-|^2 \right] \qquad \text{(dependent on } Z_{on}\text{)}$$

If we define a new set of incident and scattered wave coefficients for port N (a_n and b_n) according to

$$a_n = \frac{V_n^+}{\sqrt{Z_{on}}} \qquad\qquad b_n = \frac{V_n^-}{\sqrt{Z_{on}}}$$

then the voltage and current at the terminal plane of the n^{th} port are

$$V_n = V_n^+ + V_n^- = \sqrt{Z_{on}}\,(a_n + b_n)$$

$$I_n = \frac{1}{Z_{on}}\,(V_n^+ - V_n^-) = \frac{1}{\sqrt{Z_{on}}}\,(a_n - b_n)$$

The power delivered to the n^{th} port in terms of the new wave coefficients is

$$\begin{aligned}
P_n &= \frac{1}{2}\,\mathrm{Re}\,[V_n I_n^*] \\
&= \frac{1}{2}\,\mathrm{Re}\left[(a_n + b_n)(a_n^* - b_n^*)\right] \\
&= \frac{1}{2}\,\mathrm{Re}\left[\,|a_n|^2 - |b_n|^2 + (a_n^* b_n - a_n b_n^*)\right] \\
&= \frac{1}{2}\,\mathrm{Re}\left[\,|a_n|^2 - |b_n|^2\right] \qquad \text{(independent of } Z_{on}\text{)}
\end{aligned}$$

The *generalized scattering matrix* [S] relates the normalized incident and scattered wave coefficients a_n and b_n.

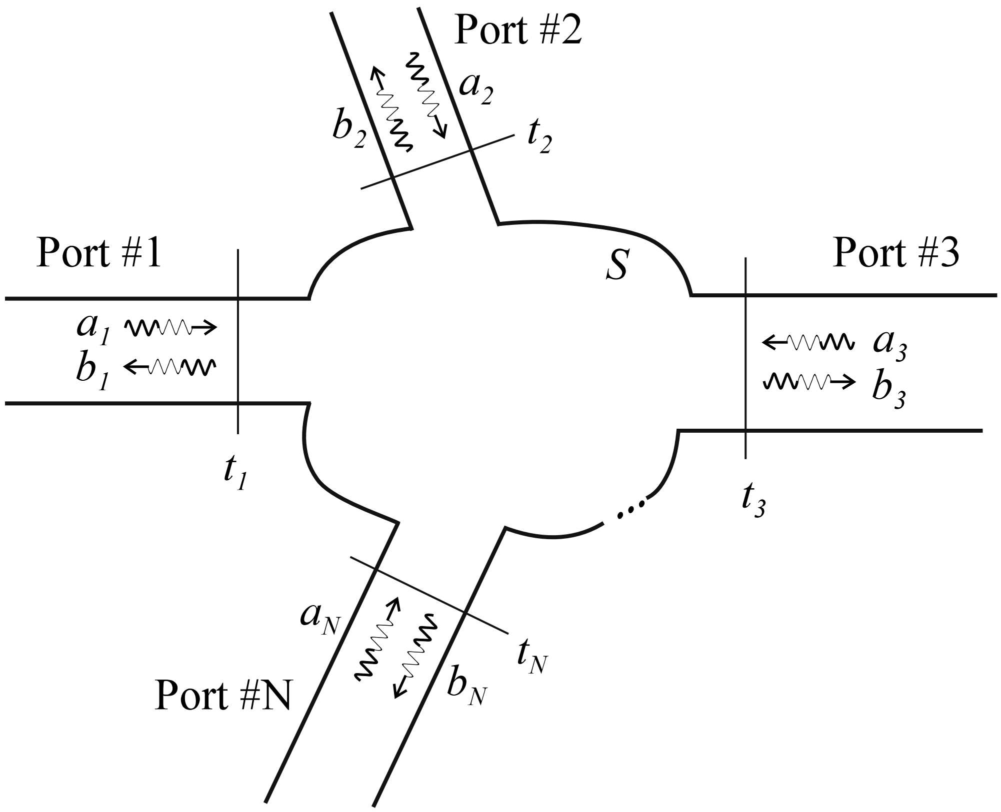
$$[b] = [S][a]$$

or

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_N \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & \cdots & S_{1N} \\ S_{21} & S_{22} & S_{23} & \cdots & S_{2N} \\ S_{31} & S_{32} & S_{33} & \cdots & S_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & S_{N3} & \cdots & S_{NN} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix}$$

where the element S_{ij} is defined as

$$S_{ij} = \frac{b_i}{a_j} \quad \text{with} \quad a_k = 0 \;\text{ for }\; k \neq j$$



The elements of the generalized scattering matrix are related to scattering matrix by

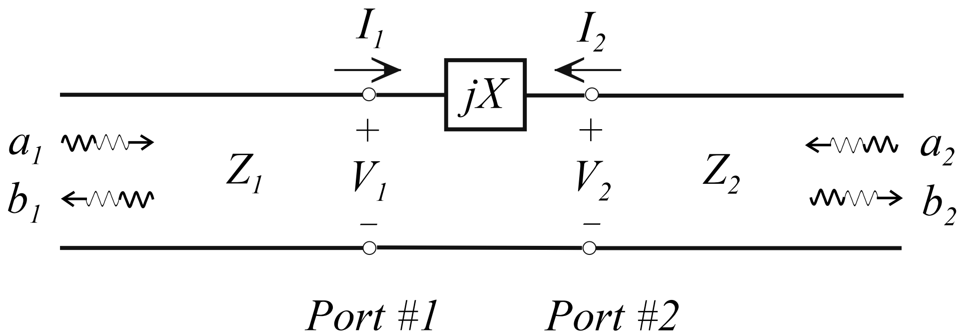
$$S_{ij} = \frac{V_i^- \sqrt{Z_{oj}}}{V_j^+ \sqrt{Z_{oi}}} \quad \text{with} \quad V_k^+ = 0 \;\text{ for }\; k \neq j$$

where

$$S_{ij} = \frac{V_i^-}{V_j^+} \quad \text{with} \quad V_k^+ = 0 \;\text{ for }\; k \neq j$$

defines the corresponding scattering matrix term.

Example (previous s-parameter example)



The scattering matrix for this example was found to be

$$[S] = \begin{bmatrix} \dfrac{Z_2 - Z_1 + jX}{Z_1 + Z_2 + jX} & \dfrac{2Z_1}{Z_1 + Z_2 + jX} \\ \dfrac{2Z_2}{Z_1 + Z_2 + jX} & \dfrac{Z_1 - Z_2 + jX}{Z_1 + Z_2 + jX} \end{bmatrix}$$

Note that the scattering matrix for this reciprocal network is not symmetric. The scattering matrix for this configuration would be symmetric if the characteristic impedances of the two transmission lines were equal. We can show that the generalized scattering matrix is symmetric for N-port networks with ports of different characteristic impedance. According to the transformation of the scattering matrix to the generalized scattering matrix:

$$S_{11} = \frac{b_1}{a_1} = \frac{V_1^-}{V_1^+} \qquad S_{12} = \frac{b_1}{a_2} = \frac{V_1^-\sqrt{Z_2}}{V_2^+\sqrt{Z_1}}$$

$$S_{21} = \frac{b_2}{a_1} = \frac{V_2^-\sqrt{Z_1}}{V_1^+\sqrt{Z_2}} \qquad S_{22} = \frac{b_2}{a_2} = \frac{V_2^-}{V_2^+}$$

$$[S] = \begin{bmatrix} \dfrac{Z_2 - Z_1 + jX}{Z_1 + Z_2 + jX} & \dfrac{2Z_1}{Z_1 + Z_2 + jX}\dfrac{\sqrt{Z_2}}{\sqrt{Z_1}} \\ \dfrac{2Z_2}{Z_1 + Z_2 + jX}\dfrac{\sqrt{Z_1}}{\sqrt{Z_2}} & \dfrac{Z_1 - Z_2 + jX}{Z_1 + Z_2 + jX} \end{bmatrix}$$

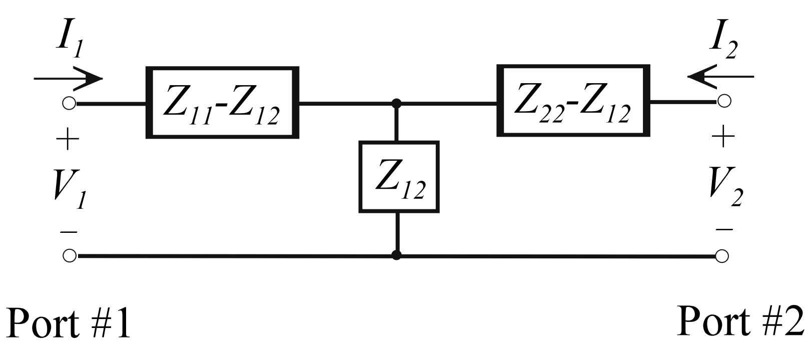
$$= \begin{bmatrix} \dfrac{Z_2 - Z_1 + jX}{Z_1 + Z_2 + jX} & \dfrac{2\sqrt{Z_1 Z_2}}{Z_1 + Z_2 + jX} \\ \dfrac{2\sqrt{Z_1 Z_2}}{Z_1 + Z_2 + jX} & \dfrac{Z_1 - Z_2 + jX}{Z_1 + Z_2 + jX} \end{bmatrix}$$

The generalized scattering matrix is symmetric (given the reciprocal network) even though the characteristic impedances of the two ports are unequal.

Equivalent Circuits for Two-Port Networks

Once the two-port parameters (*Z, Y, S, T*) for a given microwave device have been determined, we need a circuit configuration which is equivalent to the defining equations of the respective two-port parameters. If the microwave device is reciprocal, the equivalent circuit can be defined in terms of six independent parameters (the real and imaginary parts of three unique matrix elements). There are an unlimited number of equivalent circuit configurations that are possible. Two commonly used configurations are the *T-network* in terms of impedance parameters and the *π-network* in terms of admittance parameters. These networks are easily implemented with any type of two-port parameters given the basic transformations among the different sets of parameters (see Table 4.2, p. 211)

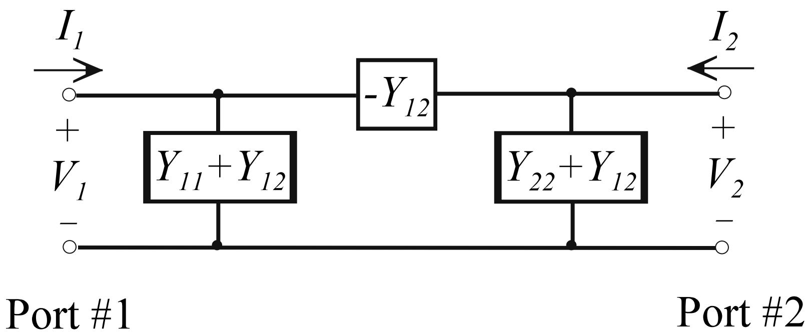
T-Network



$$V_1 = I_1(Z_{11} - Z_{12}) + (I_1 + I_2)Z_{12} = Z_{11}I_1 + Z_{12}I_2$$

$$V_2 = I_2(Z_{22} - Z_{12}) + (I_1 + I_2)Z_{12} = Z_{12}I_1 + Z_{22}I_2$$

π-Network



$$I_1 = V_1(Y_{11} + Y_{12}) + (V_1 - V_2)(-Y_{12}) = Y_{11}V_1 + Y_{12}V_2$$

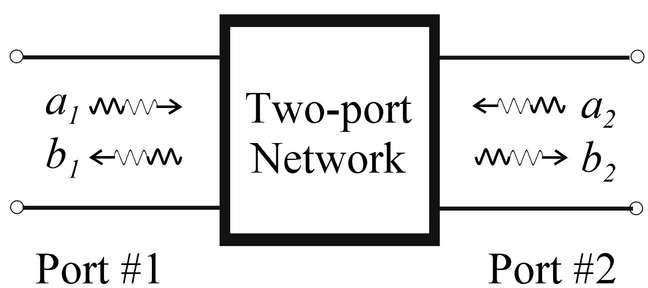
$$I_2 = V_2(Y_{22} + Y_{12}) + (V_1 - V_2)(-Y_{12}) = Y_{12}V_1 + Y_{22}V_2$$

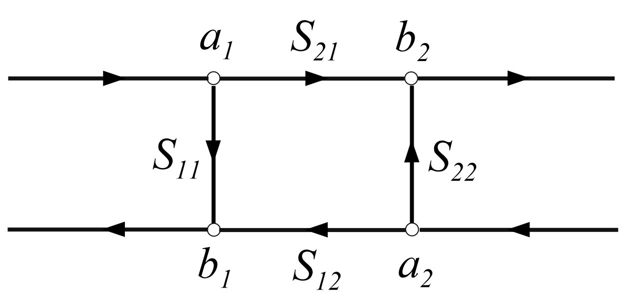
Signal Flow Graphs

Signal flow graphs are a graphical technique of analyzing microwave networks in terms of the incident and scattered (transmitted and reflected) waves at the network ports. Signal flow graphs consist of *nodes* and *branches* which may be related directly to the generalized scattering parameters.

Nodes - Each node represents either an incident wave (a wave entering the port) or a scattered wave (a wave leaving the port). Thus, the signal flow graph for a an N-port network contains 2N nodes. Following the convention of the generalized scattering parameters, the i^{th} port is defined by two nodes: a_i represents the wave entering the i^{th} port while b_i represents the wave leaving the i^{th} port.

Branches - Each branch is a path from an *a*-node to a *b*-node which represents the signal flow within the N-port network. Thus, each branch is associated with a particular scattering parameter or a reflection coefficient.



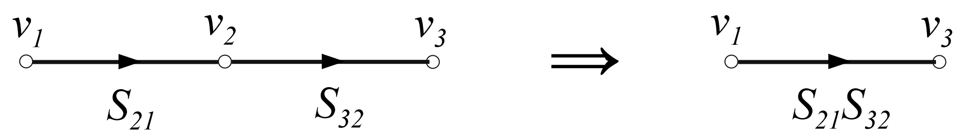


$$b_1 = S_{11}a_1 + S_{12}a_2$$

$$b_2 = S_{21}a_1 + S_{22}a_2$$

Decomposition Rules for Signal Flow Graphs

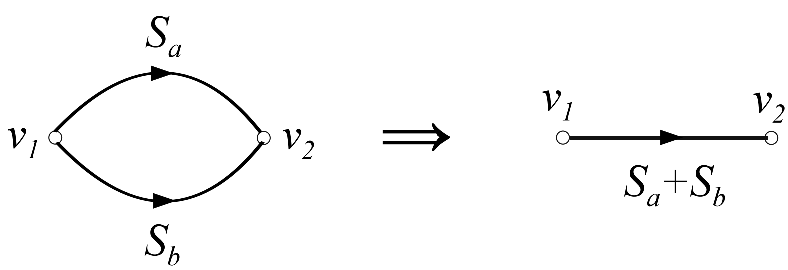
Series Rule



$$v_2 = S_{21} v_1$$

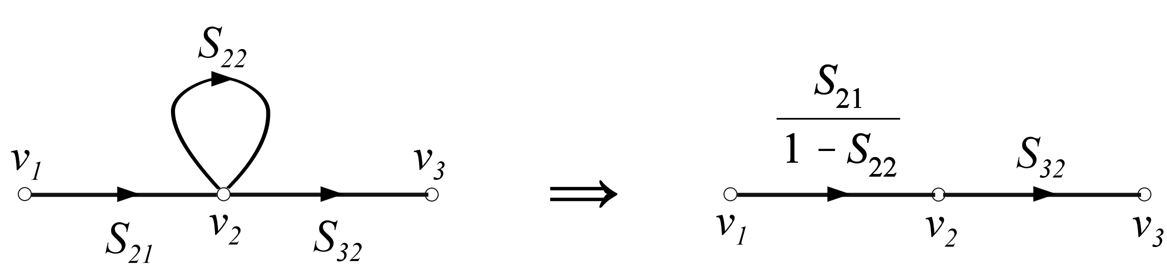
$$v_3 = S_{32} v_2 = S_{21} S_{32} v_1$$

Parallel Rule



$$v_2 = S_a v_1 + S_b v_1 = (S_a + S_b) v_1$$

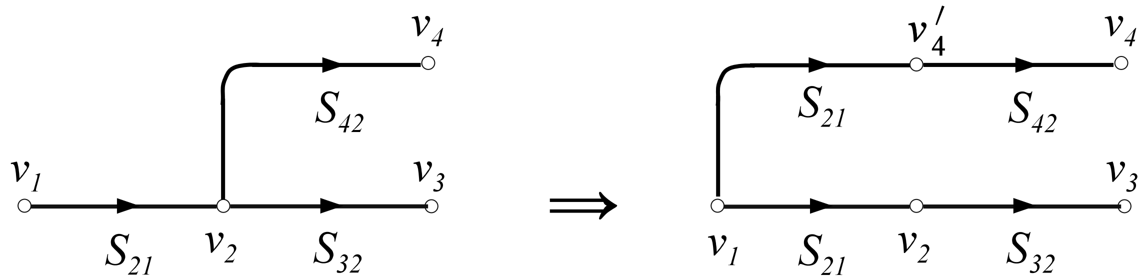
Self-loop Rule



$$v_2 = S_{21} v_1 + S_{22} v_2 \quad \Rightarrow \quad v_2 = \frac{S_{21}}{1 - S_{22}} v_1$$

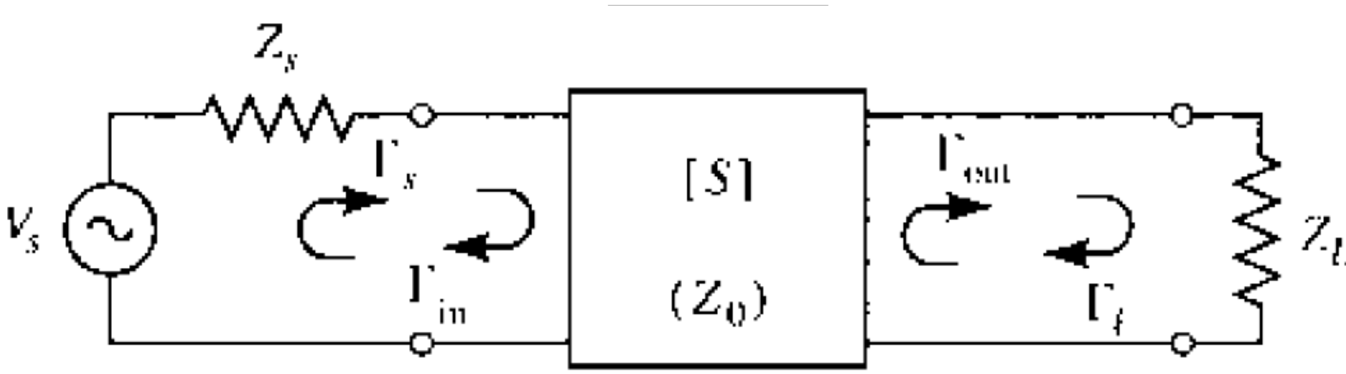
$$v_3 = S_{32} v_2$$

Splitting Rule



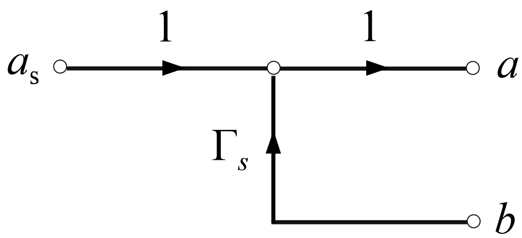
$$v_4 = S_{42} v_2 = S_{21} S_{42} v_1$$

Example (Signal flow graph) Determine the input reflection coefficient (Γ_{in}) for the terminated two-port network shown below using a signal flow graph.



The input reflection coefficient may be determined by manipulating the signal flow graph for the terminated two-port network. We need signal graph models for the connection of the source to the two-port and the connection of the load to the two-port.

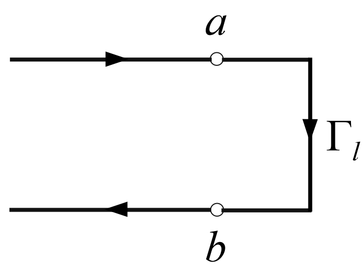
The signal flow graph for the source in the terminated two-port network must include reflections due to mismatch. The signal flow graph model for the input source (shown below) accounts for scattered waves from the two port network that may be reflected back to the network through the reflection coefficient for the source, Γ_s. Note that the input voltage has been normalized according to the scattering parameter definition.



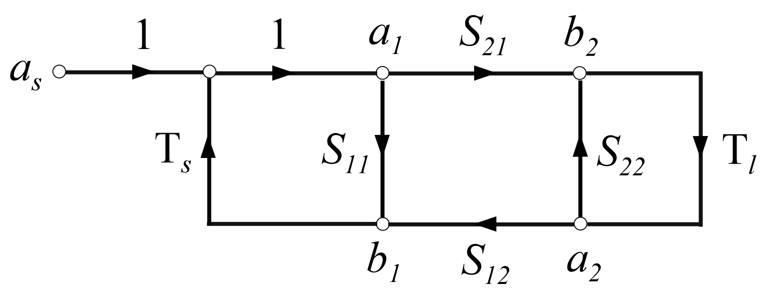
$$a_s = \frac{V_s}{\sqrt{Z_o}}$$

$$a = a_s + \Gamma_s b$$

The signal flow graph for the termination accounts for reflections due to mismatch through the load reflection coefficient, Γ_l.



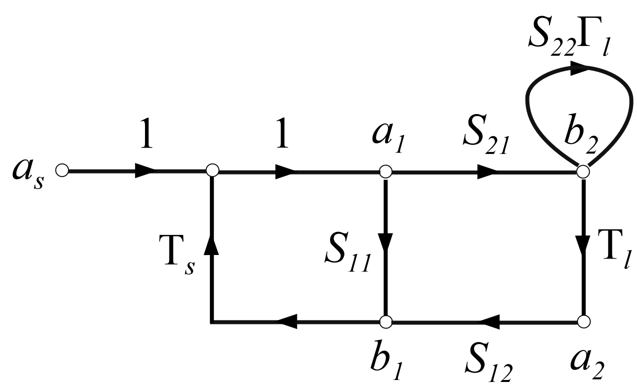
Combining the signal flow graphs of the source, the load and the two-port network yields the signal flow graph for the complete circuit.



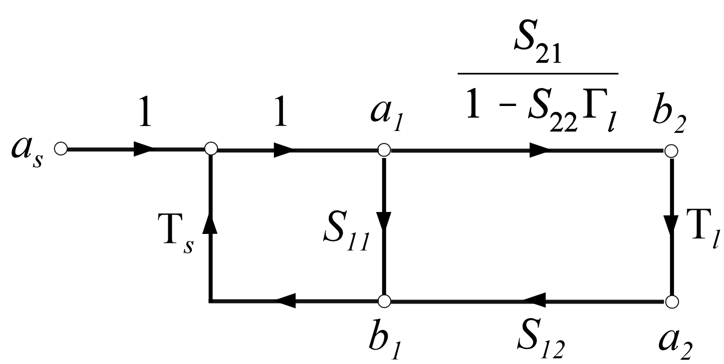
We may use the splitting rule on node a_2. The path directly from a_2 to b_2 can be transformed into a self-loop by noting that

$$a_2 = \Gamma_l b_2$$

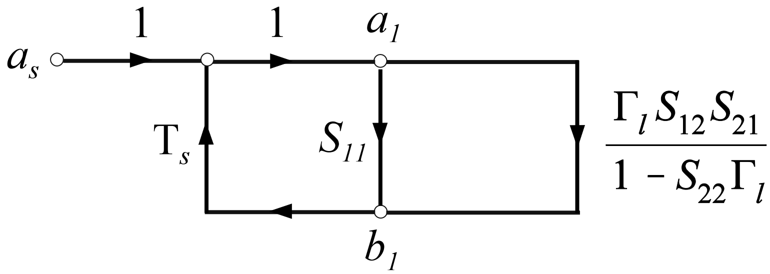
$$b_2 = S_{21} a_1 + S_{22} a_2 = S_{21} a_1 + S_{22} \Gamma_l b_2$$



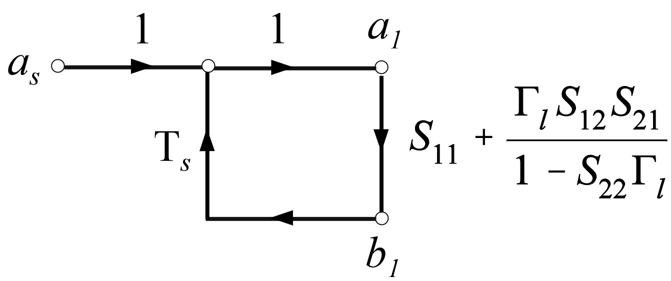
Node b_2 can then be transformed using the self-loop rule.



The path from node a_1 to b_2 to a_2 to b_1 can the be combined into a single path using the series rule.



The two paths between nodes a_1 and b_1 can be combined using the parallel rule.



The single path from node a_1 to b_1 allows us to write the input reflection coefficient directly from the reduced signal flow graph.

$$\Gamma_{in} = S_{11} + \frac{\Gamma_l S_{12} S_{21}}{1 - S_{22}\Gamma_l}$$