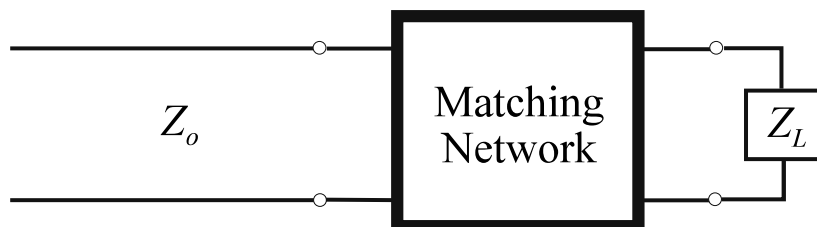


Impedance Matching and Transformation

Matching the source and load to the transmission line or waveguide in a general microwave network is necessary to deliver maximum power from the source to the load. In many cases, it is not possible to choose all impedances such that overall matched conditions result. These situations require that matching networks be used to eliminate reflections.

T-line or waveguide to termination matching network



T-line or waveguide to t-line or waveguide matching network



Depending on the application, matching may be required over a band of frequencies such that the bandwidth of the matching network is an important design parameter. If the load impedance varies over a given range, a matching network which can be adjusted or tuned as necessary. In general, matching networks are constructed with reactive components only so that no loss is added to the overall network.

Lumped Element Networks

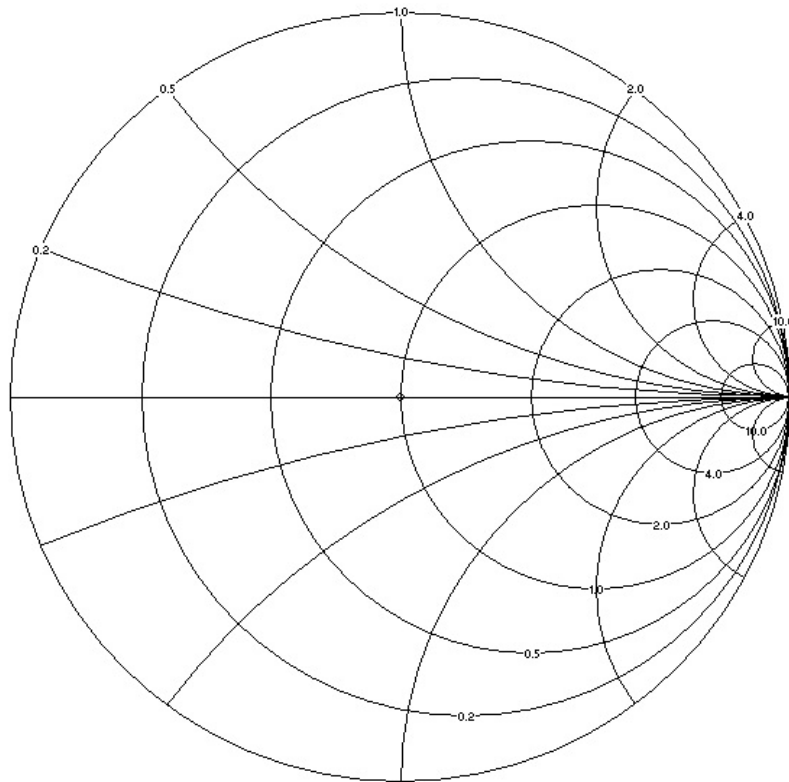
For frequencies up to approximately 1 GHz, matching networks containing lumped elements (*L-networks*) may be used. The circuit elements (capacitors and inductors) must be small enough relative to wavelength so that the normal circuit equations for voltage and current are valid. L-networks are easily analyzed using either circuit equations or the Smith chart.

The configuration of the matching L-network will depend on the size of the load impedance relative to the characteristic impedance Z_o . The general load impedance Z_L and its corresponding normalized value z_L defined by

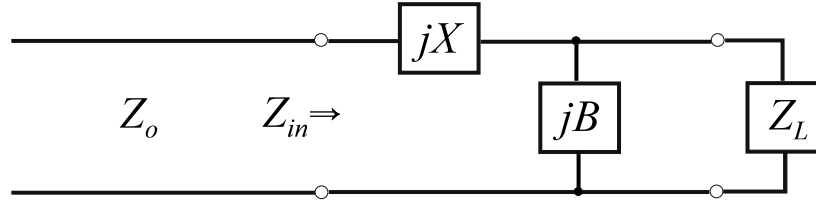
$$Z_L = R_L + jX_L \quad \Rightarrow \quad z_L = \frac{Z_L}{Z_o} = \frac{R_L}{Z_o} + j\frac{X_L}{Z_o} = r_L + jx_L$$

If $R_L < Z_o$, then $r_L < 1$ (z_L is outside the $r=1$ circle on the Smith chart).

If $R_L > Z_o$, then $r_L > 1$ (z_L is inside the $r=1$ circle on the Smith chart).



Lumped Element Matching Network for $R_L > Z_o$



The reactance jX and susceptance jB may be any combination of capacitors ($X < 0, B > 0$) or inductors ($X > 0, B < 0$). For a matched network, the input impedance Z_{in} must be equal to Z_o which gives

$$Z_{in} = Z_o = jX + \left(jB + \frac{1}{Z_L} \right)^{-1} = jX + \frac{1}{\left(jB + \frac{1}{R_L + jX_L} \right)}$$

$$Z_o = jX + \frac{R_L + jX_L}{jBR_L - BX_L + 1}$$

Multiplying out this equation and equating the real and imaginary terms on both sides of the equation yields two equations for the unknowns X and B .

$$B(XR_L - X_L Z_o) = R_L - Z_o \quad (1)$$

$$X(1 - BX_L) = BZ_o R_L - X_L \quad (2)$$

If we solve Equation (1) for X , and insert the result into (2), we find a quadratic equation for B with a solution of

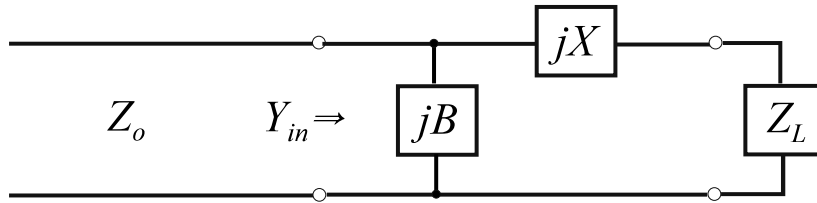
$$B = \frac{X_L \pm \sqrt{\frac{R_L}{Z_o} \sqrt{R_L^2 + X_L^2 - Z_o R_L}}}{R_L^2 + X_L^2}$$

The requirement that $R_L > Z_o$ ensures that the term under the square root in the numerator of the expression for B is real. Note that two solutions for B are possible and both solutions are physically realizable given that B can be positive or negative. Once B is determined, X can be found using Equation (2):

$$X = \frac{BZ_oR_L - X_L}{1 - BX_L}$$

With two pairs of solutions for both B and X , there are two different matching network solutions.

Lumped Element Matching Network for $R_L < Z_o$



For a matched network, the input admittance Y_{in} must be equal to $1/Z_o$ which gives

$$Y_{in} = \frac{1}{Z_o} = jB + \frac{1}{R_L + j(X + jX_L)}$$

Multiplying out this equation and equating the real and imaginary terms on both sides of the equation yields two equations for the unknowns X and B .

$$BZ_o(X + X_L) = Z_o - R_L \quad (1)$$

$$(X + X_L) = BZ_oR_L \quad (2)$$

Solving these equations for B and X yields

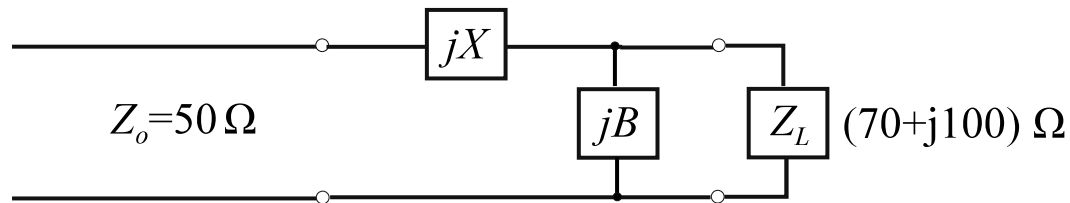
$$B = \pm \frac{1}{Z_o} \sqrt{\frac{(Z_o - R_L)}{R_L}}$$
$$X = \pm \sqrt{R_L(Z_o - R_L)} - X_L$$

The requirement that $R_L < Z_o$ ensures that the terms under the square roots in the expressions for B and X are real. Again, with two solution pairs for B and X , there are two different matching network solutions.

Example (Lumped element matching networks)

Design two lumped element matching networks to match a $50\ \Omega$ line to a load impedance of $Z_L = (70 + j\ 100)\ \Omega$ [$R_L > Z_o$] at 700 MHz..

$$z_L = \frac{70 + j100}{50} = 1.4 + j2.0 = r_L + jx_L$$



Using the matching network design equations, we find

$$B_1 = 1.52 \times 10^{-2}, \quad B_2 = -1.77 \times 10^{-3}\ S$$

$$X_1 = 90, \quad X_2 = -90\ \Omega$$

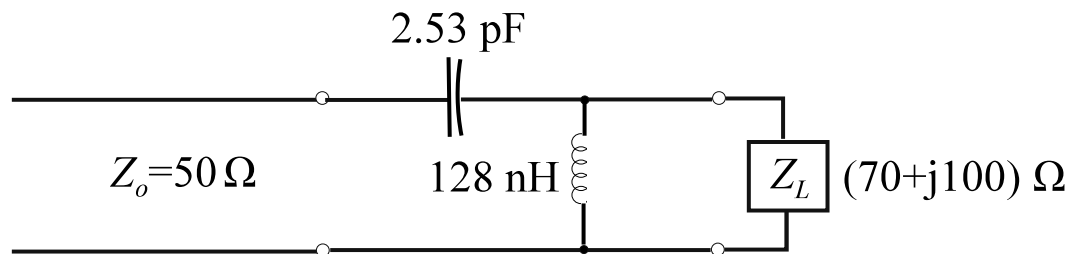
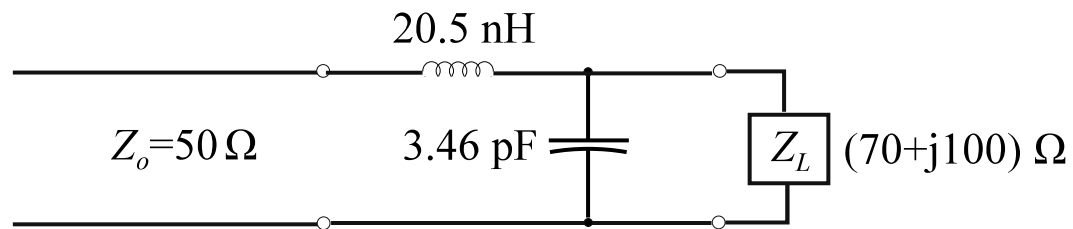
These solutions correspond to lumped elements of

$$jB_1 = j1.52 \times 10^{-2} = j\omega C_1 \quad \Rightarrow \quad C_1 = \frac{1.52 \times 10^{-2}}{2\pi 700 \times 10^6} = 3.46\ \text{pF}$$

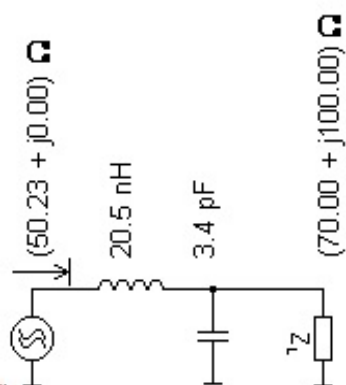
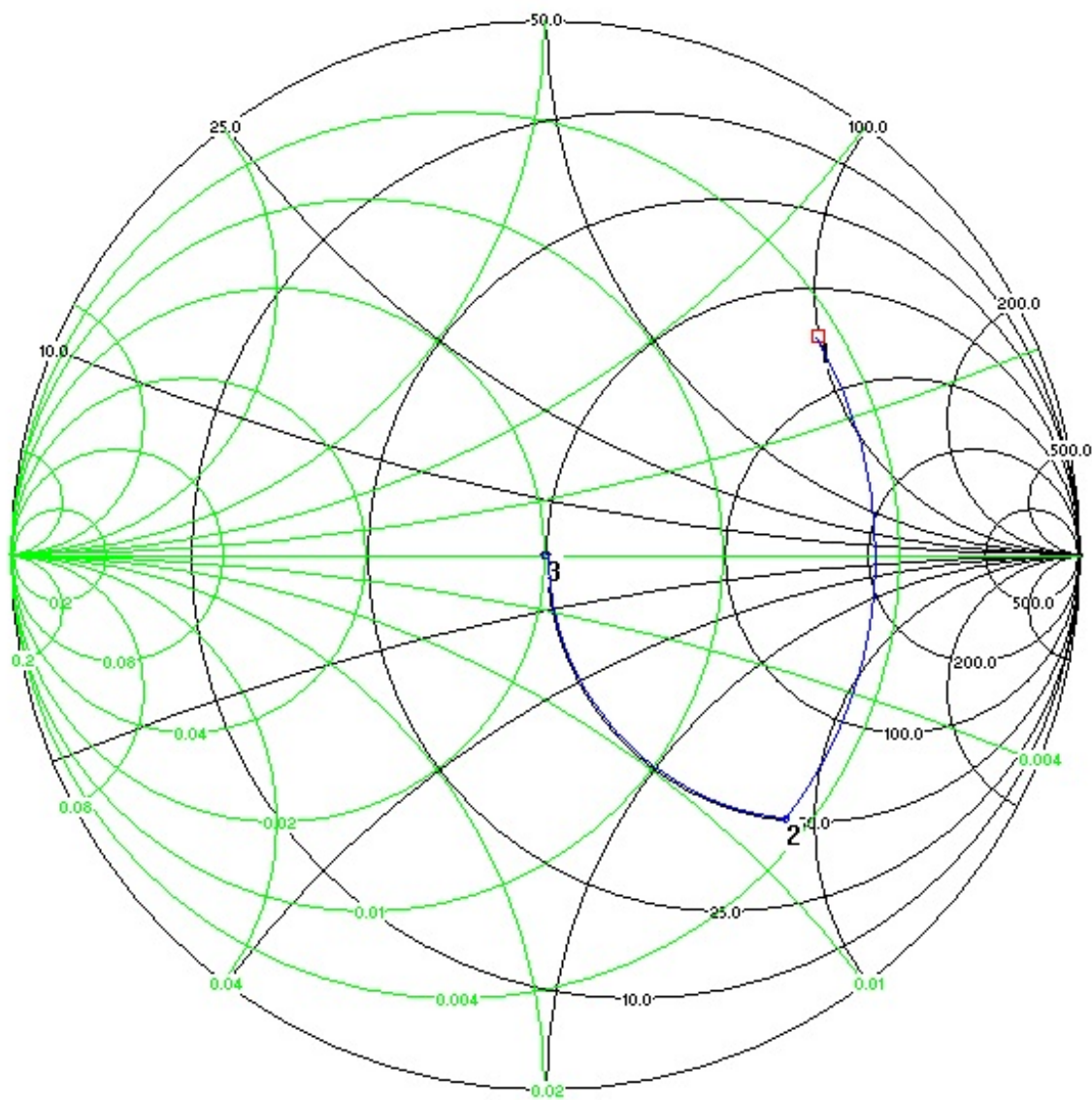
$$jX_1 = j90 = j\omega L_1 \quad \Rightarrow \quad L_1 = \frac{90}{2\pi 700 \times 10^6} = 20.5\ \text{nH}$$

$$jB_2 = -j1.77 \times 10^{-3} = \frac{1}{j\omega L_2} \quad \Rightarrow \quad L_2 = \frac{1}{2\pi 700 \times 10^6 (1.77 \times 10^{-3})} \\ = 128\ \text{nH}$$

$$jX_2 = -j90 = \frac{1}{j\omega C_2} \quad \Rightarrow \quad C_2 = \frac{1}{2\pi 700 \times 10^6 (90)} = 2.53\ \text{pF}$$

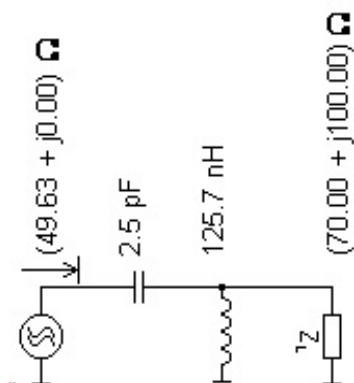
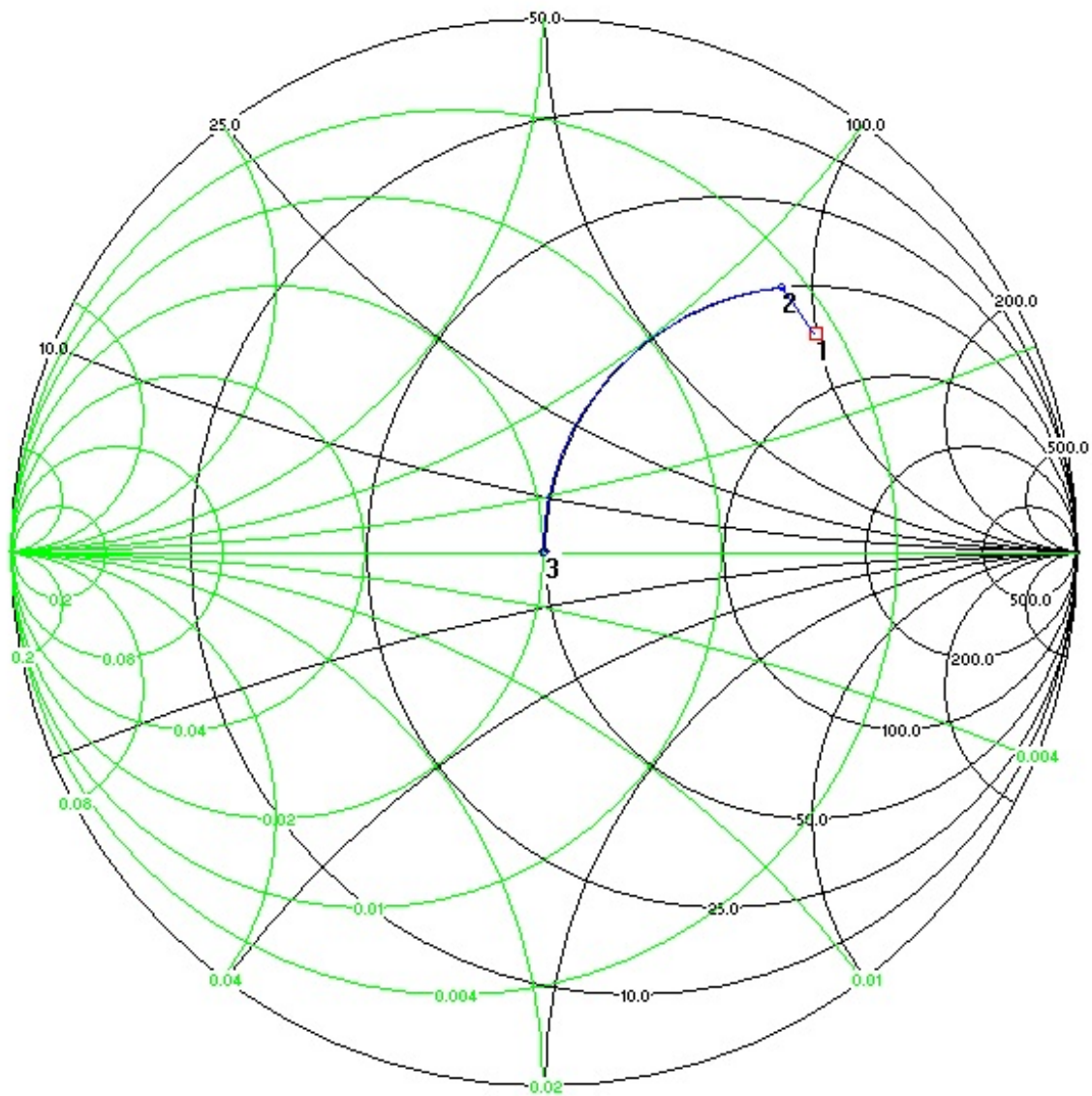


The Smith chart can also be used to design the matching networks. We first locate the load impedance on the Smith chart. Given the parallel connection of the rightmost matching network element (jB) with the load, we add the admittance of these two elements together. Since the parallel matching network element is purely susceptive, we move along the constant conductance circle from the load admittance in the proper direction for the given matching network element (smaller admittance if jB represents an inductor or larger admittance if jB represents a capacitor). Using admittances, we rotate until we intersect the $1/Z_o + j0$ admittance circle (or the Z_o impedance circle using impedances). The change in the susceptance of these two points represents the matching element jB . The impedance of the parallel combination of the load and jB is then added to the reactance of the matching element jX by rotating along the $Z_o + j0$ impedance circle until we reach the center of the Smith chart (matched condition).



$Z_0 = 50.00 \, \Omega$
 $f_{\text{main}} = 700.000 \text{ MHz}$

Data Point #1 $(70.0 + j100.0) \, \Omega$
 Data Point #2 $(50.2 - j90.4) \, \Omega$
 Data Point #3 $(50.2 + j0.0) \, \Omega$



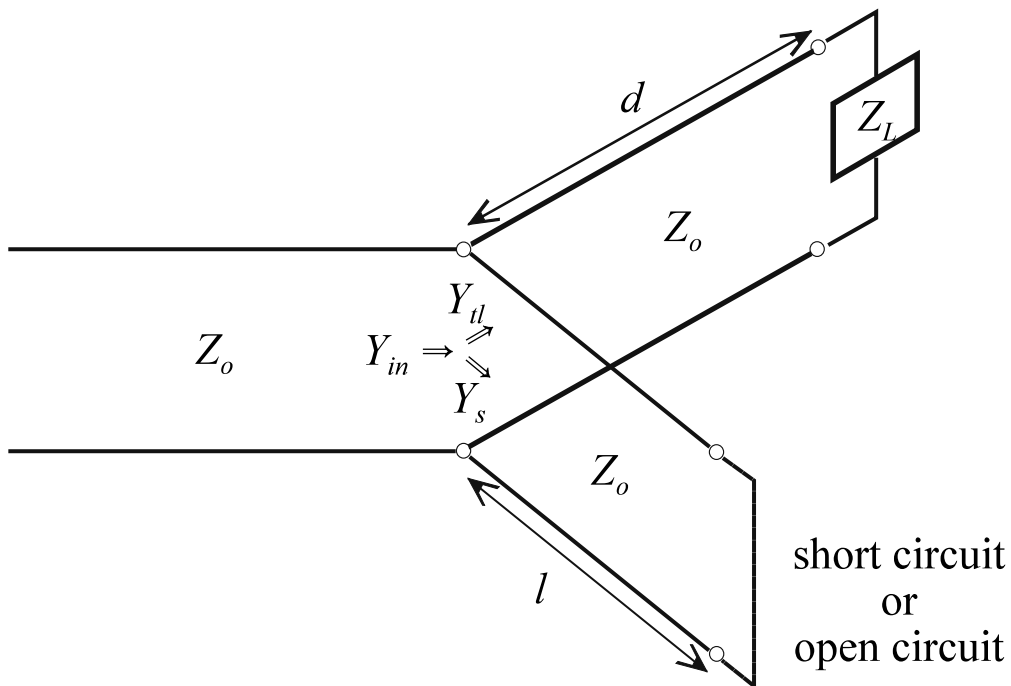
$Z_0 = 50.00 \Omega$
 $f_{\text{main}} = 700.000 \text{ MHz}$

Data Point #1 $(70.0 + j100.0) \Omega$
 Data Point #2 $(49.6 + j90.0) \Omega$
 Data Point #3 $(49.6 + j0.0) \Omega$

Single Stub Tuners

Given that we can obtain any value of reactance or susceptance with the proper length of short-circuited or open-circuited transmission line, we may use these transmission line stubs as matching networks.

Shunt Stub



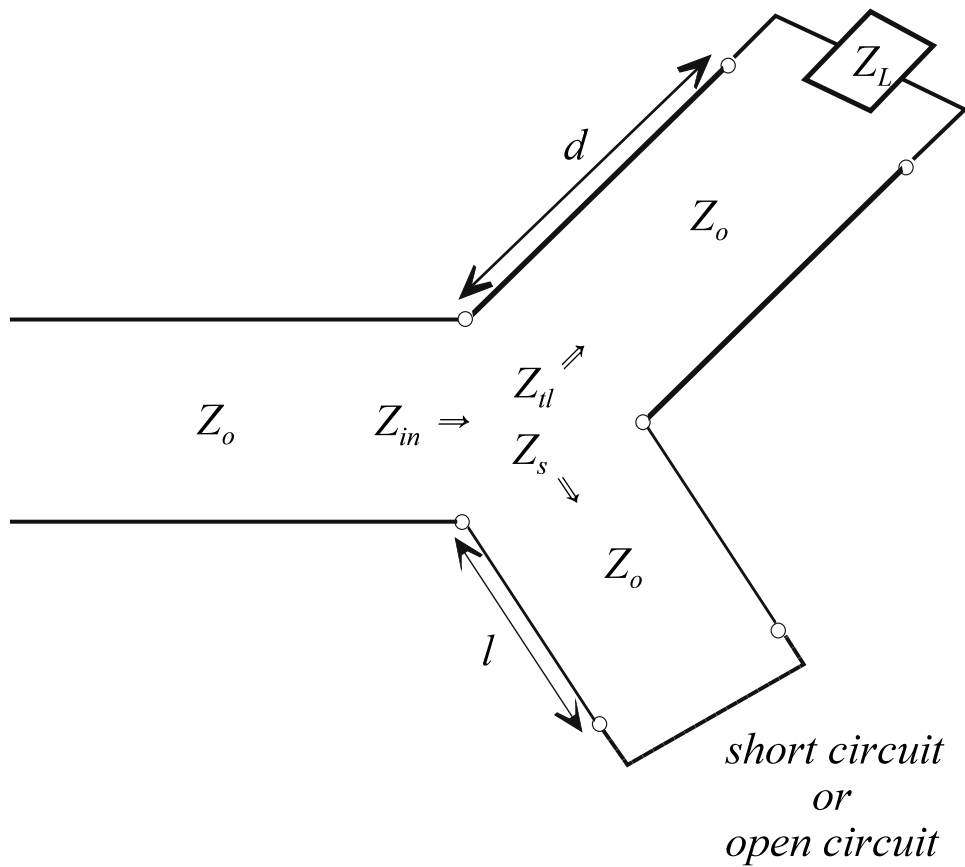
$$Y_{tl} = Y_o + jB \quad [\text{Input admittance of the terminated t-line section}]$$

$$Y_s = -jB \quad [\text{Input admittance of the stub (short or open circuit)}]$$

$$Y_{in} = Y_{tl} + Y_s = Y_o \quad [\text{Overall input admittance}]$$

$$Y_o = \frac{1}{Z_o}$$

Series Stub



$$Z_{tl} = Z_o + jX \quad [\text{Input impedance of the terminated t-line section}]$$

$$Z_s = -jX \quad [\text{Input impedance of the stub (short or open circuit)}]$$

$$Z_{in} = Z_{tl} + Z_s = Z_o \quad [\text{Overall input impedance}]$$

Single Shunt Stub Tuner Design Procedure

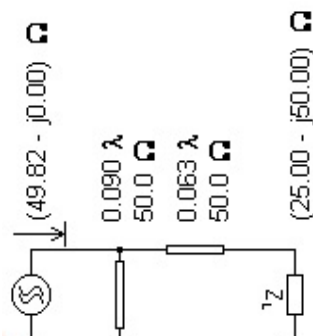
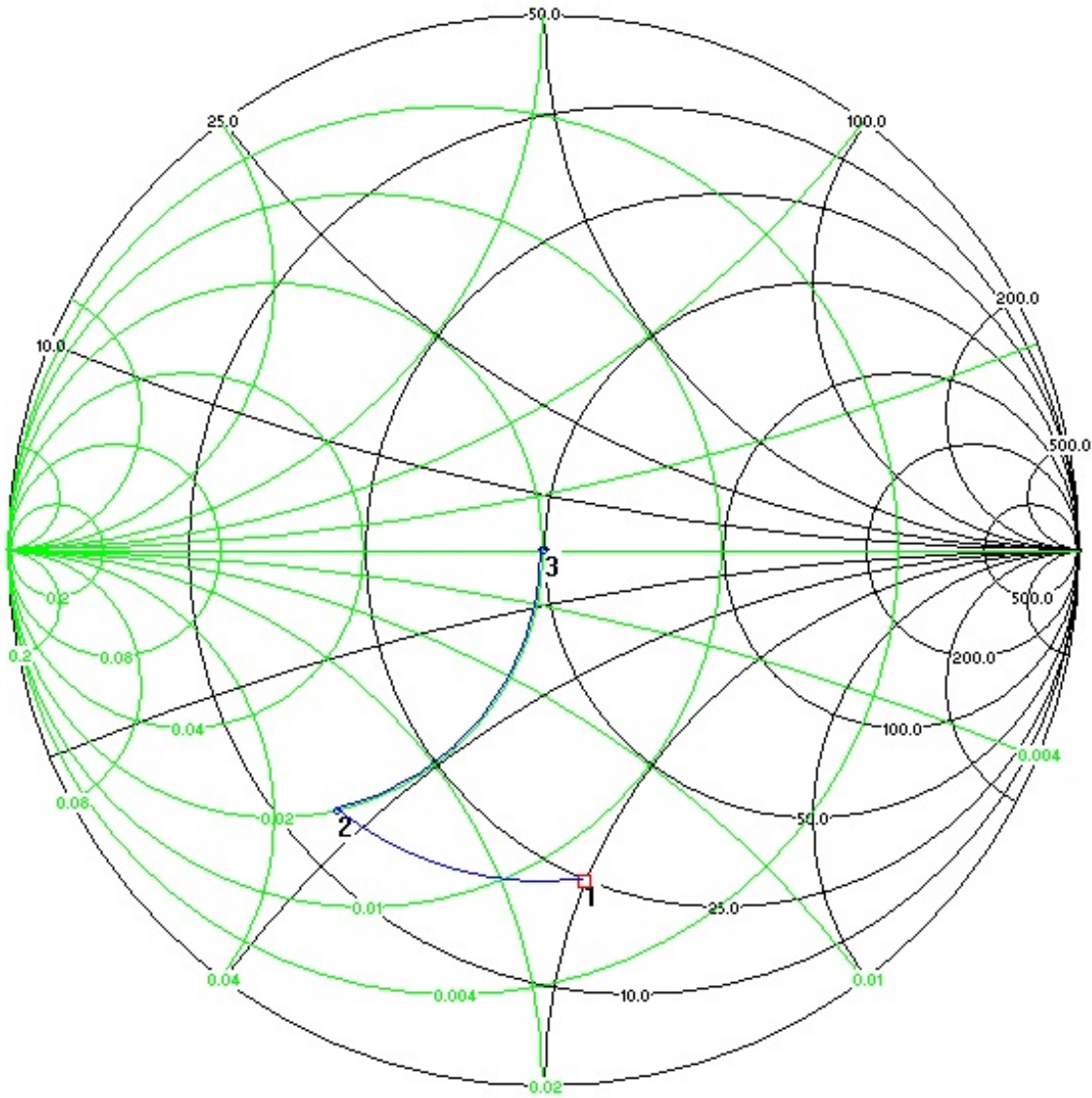
1. Locate normalized load impedance and draw VSWR circle (normalized load admittance point is 180° from the normalized impedance point).
2. From the normalized load admittance point, rotate CW (toward generator) on the VSWR circle until it intersects the $r = 1$ circle. This rotation distance is the length d of the terminated section of t-tline. The normalized admittance at this point is $1 + jb$.
3. Beginning at the stub end (rightmost Smith chart point is the admittance of a short-circuit, leftmost Smith chart point is the admittance of an open-circuit), rotate CW (toward generator) until the point at $0 - jb$ is reached. This rotation distance is the stub length l .

Single Series Stub Tuner Design Procedure

1. Locate normalized load impedance and draw VSWR circle.
2. From the normalized load impedance point, rotate CW (toward generator) on the VSWR circle until it intersects the $r = 1$ circle. This rotation distance is the length d of the terminated section of t-tline. The normalized impedance at this point is $1 + jx$.
3. Beginning at the stub end (leftmost Smith chart point is the impedance of a short-circuit, rightmost Smith chart point is the impedance of an open-circuit), rotate CW (toward generator) until the point at $0 - jx$ is reached. This rotation distance is the stub length l .

Example (shunt stub tuner)

Design a short-circuited shunt stub tuner to match a load impedance of $Z_L = (25 - j50) \Omega$ to a 50Ω transmission line.

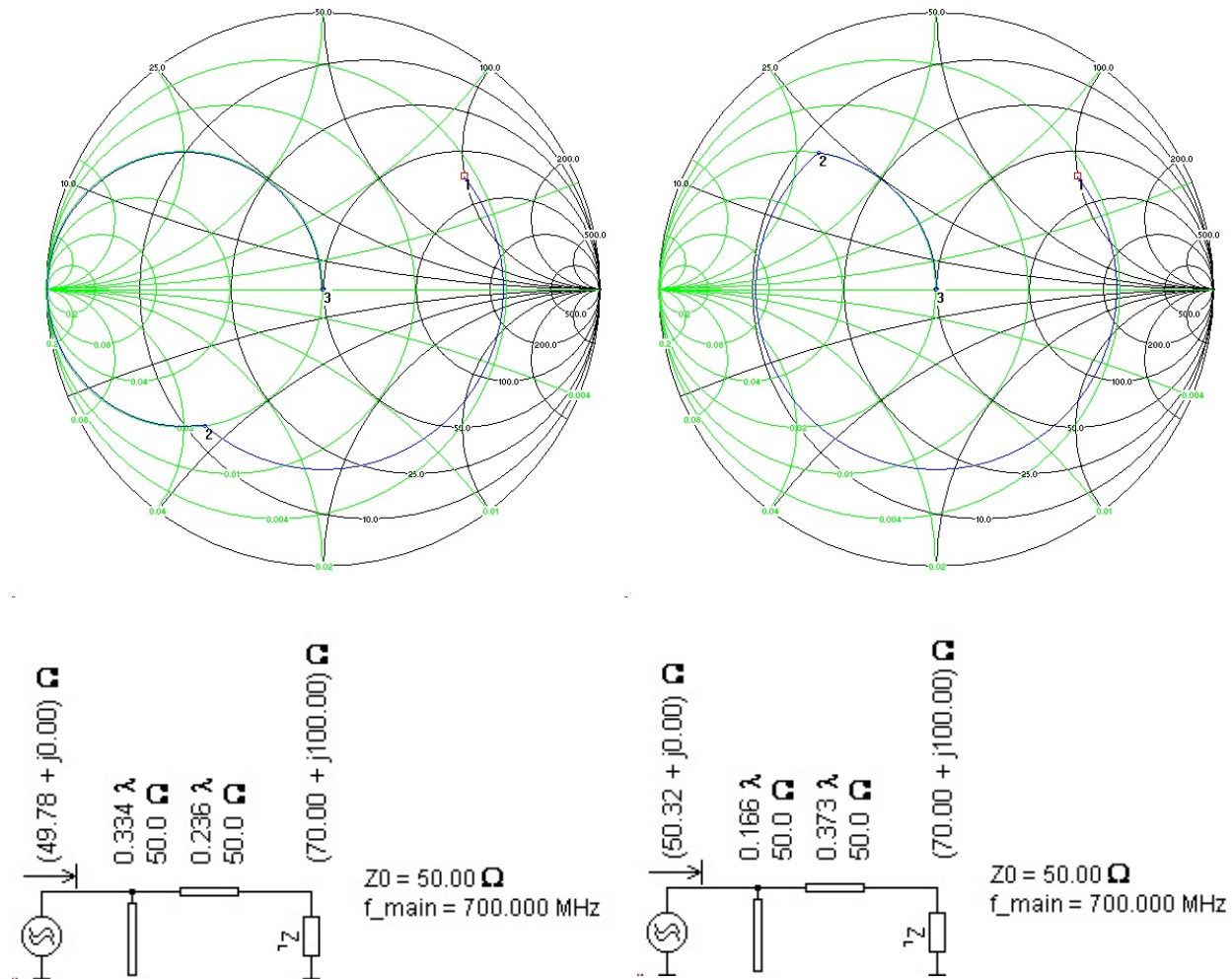


$Z_0 = 50.00 \Omega$
 $f_{\text{main}} = 500.000 \text{ MHz}$

Data Point #1 $(25.0 - j50.0) \Omega$
Data Point #2 $(14.3 - j22.5) \Omega$
Data Point #3 $(49.8 - j0.0) \Omega$

Example (open-circuited shunt stub tuner)

Design two open-circuited shunt stub tuners to match the load impedance in the previous lumped element matching network example, $[Z_L = (70+j100) \Omega, 50 \Omega$ transmission line at 700 MHz]. The two designs should represent the two shortest stub distances from the load.



Data Point #1 $(70.0 + j100.0) \Omega$
 Data Point #2 $(12.6 - j21.6) \Omega$
 Data Point #3 $(49.8 + j0.0) \Omega$

$$d_1 = 101 \text{ mm}$$

$$l_1 = 143 \text{ mm}$$

Data Point #1 $(70.0 + j100.0) \Omega$
 Data Point #2 $(12.6 + j21.8) \Omega$
 Data Point #3 $(50.3 + j0.0) \Omega$

$$d_2 = 160 \text{ mm}$$

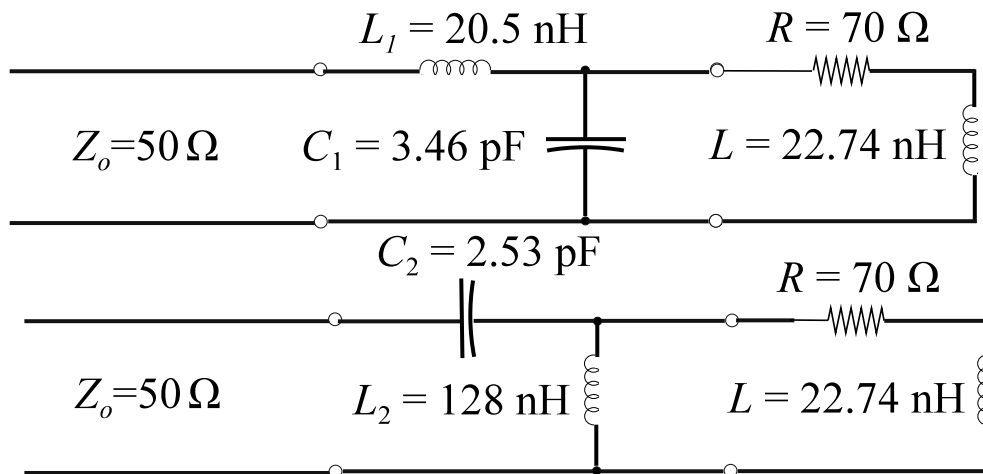
$$l_2 = 71 \text{ mm}$$

Frequency Response of Matching Networks

Ideal lumped element and single stub matching networks provide perfect matching ($\Gamma=0$) at only one frequency. In our lumped element and single stub matching network examples, we showed two solutions that yielded perfect matching at the design frequency. However, the component configuration in a lumped element matching network and the stub position in a stub matching network will affect the frequency response of the network away from the design frequency. We may plot the frequency response of the reflection coefficient to illustrate the different responses. Given either type of matching network, the reflection coefficient looking into the matching network may be written as

$$\Gamma(\omega) = \frac{Z_{in}(\omega) - Z_o}{Z_{in}(\omega) + Z_o}$$

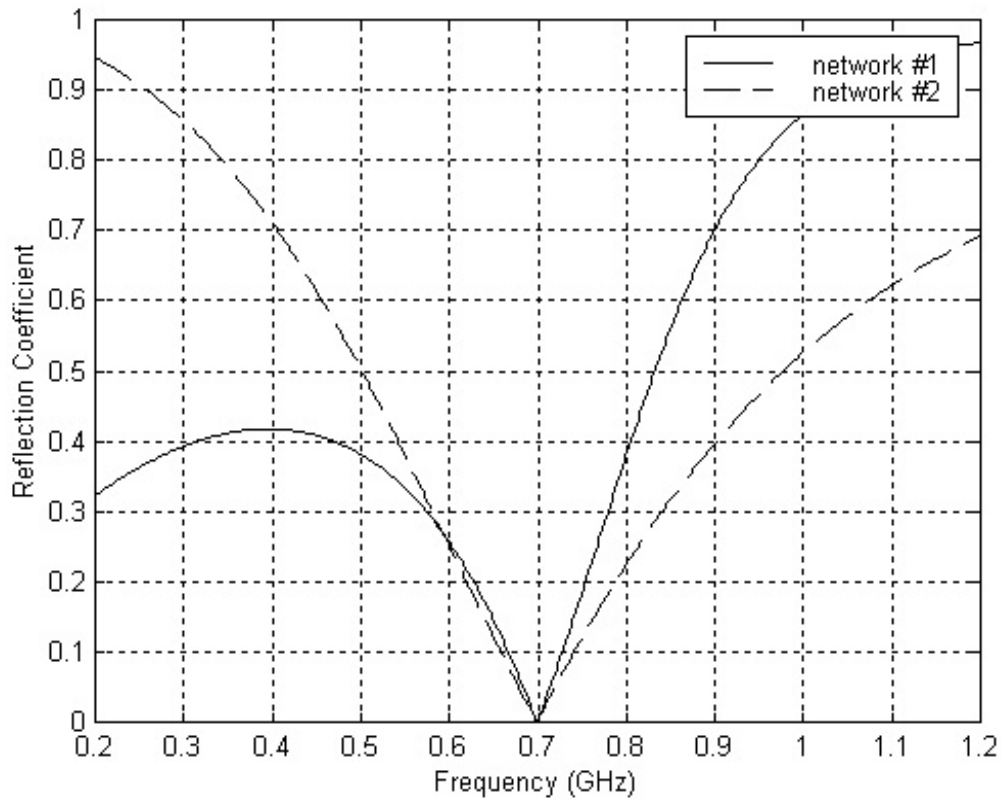
In order to determine the variation of Z_{in} with respect to frequency, we need to know the variation of the load impedance with respect to frequency. We assume that the load impedance used in our examples ($Z_L = 70 + j100 \Omega$ at 700 MHz) consists of a series combination of a resistor ($R = 70 \Omega$) and an inductor ($L = 22.74 \text{ nH}$) [$\omega L = 100$ at $f = 700 \text{ MHz}$]. The resulting two circuits for the lumped element matching networks are shown below.



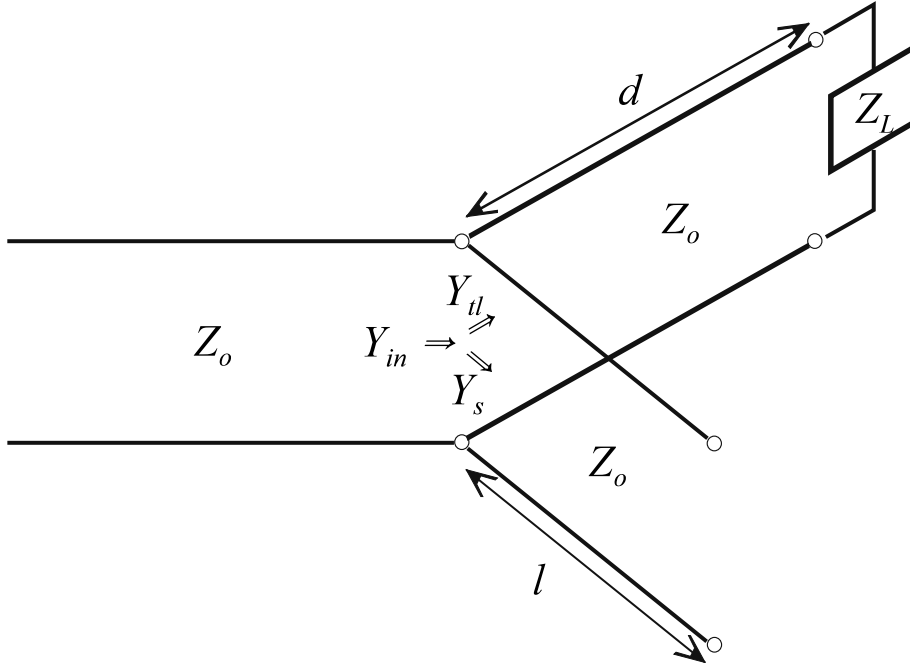
According to the design equations, the input impedances for the two networks looking into the matching network input ports are

$$Z_{in,1} = jX_1 + \frac{1}{\left(jB_1 + \frac{1}{R_L + jX_L} \right)} = j\omega L_1 + \frac{1}{\left(j\omega C_1 + \frac{1}{R + j\omega L} \right)}$$

$$Z_{in,2} = jX_2 + \frac{1}{\left(jB_2 + \frac{1}{R_L + jX_L} \right)} = \frac{1}{j\omega C_2} + \frac{1}{\left(j\omega L_2 + \frac{1}{R + j\omega L} \right)}$$



For the case of the shunt stub networks, the input admittance looking into the matching network is



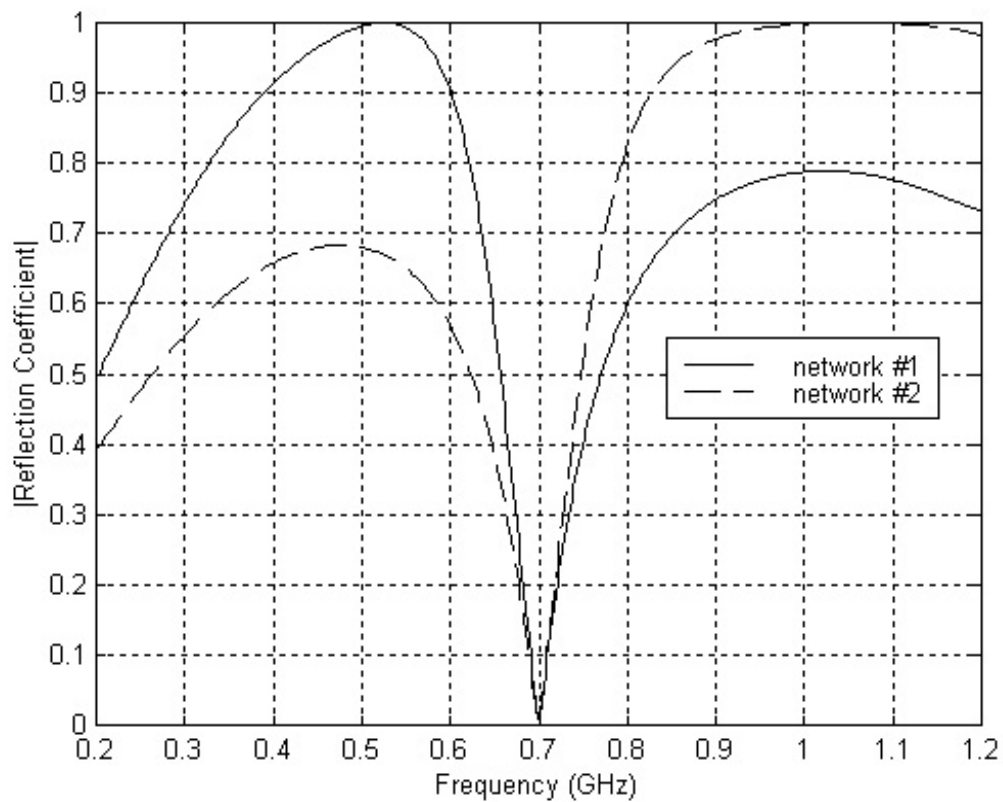
$$Y_{in} = Y_{tl} + Y_s$$

$$Y_s = \frac{1}{Z_s} = \frac{1}{-jZ_o \cot \beta l} = j \frac{\tan \beta l}{Z_o}$$

$$Y_{tl} = \frac{1}{Z_{tl}} = \frac{1}{Z_o} \frac{Z_o + jZ_L \tan \beta d}{Z_L + jZ_o \tan \beta d}$$

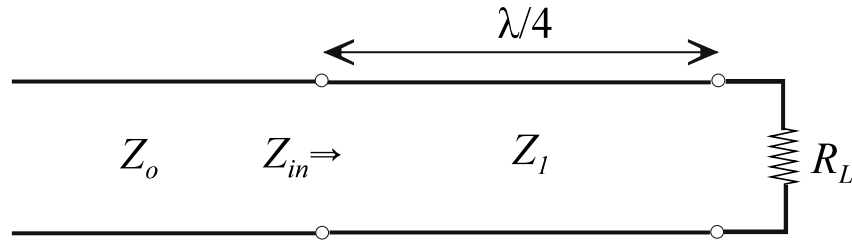
$$Z_{in} = \frac{1}{Y_{in}}$$

We find $Z_{in,1}$ by inserting (l_1, d_1) and find $Z_{in,2}$ by inserting (l_2, d_2) .



Comparing the frequency responses of the lumped element matching networks and the stub tuners shows that, in general, the lumped elements yield a slightly broader bandwidth.

Quarter Wave Transformer



The quarter wave transformer is a simple quarter wavelength section of transmission line with characteristic impedance Z_1 that when placed between a transmission line of characteristic impedance Z_o and a real load impedance R_L yields a matched system. The value of Z_1 is determined by using the equation for the input impedance of a terminated transmission line.

$$Z_{in} = Z_1 \frac{R_L + jZ_1 \tan \beta l}{Z_1 + jR_L \tan \beta l}$$

The input impedance is purely real since the line length is one quarter wavelength:

$$\beta l = \frac{2\pi}{\lambda} \frac{\lambda}{4} = \frac{\pi}{2}$$

The tangent terms become unbounded so that, in the limit, we find

$$Z_{in} = \frac{Z_1^2}{R_L}$$

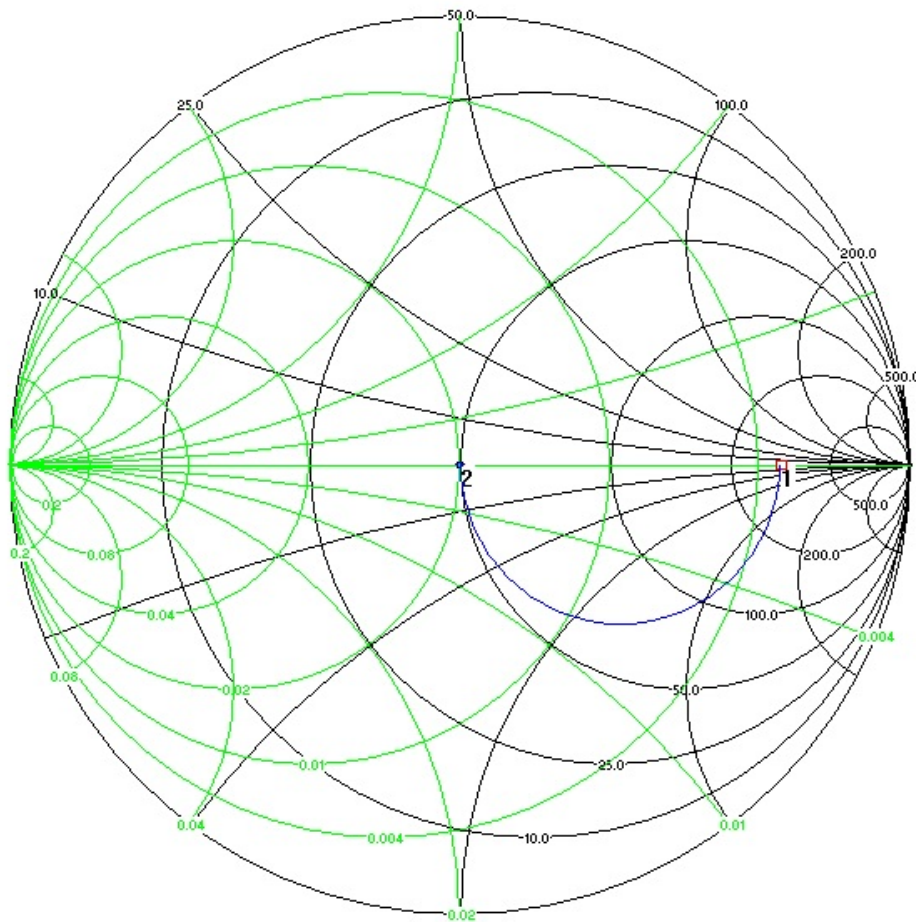
For a matched system, the input impedance must equal Z_o which gives

$$Z_{in} = \frac{Z_1^2}{R_L} = Z_o \quad \Rightarrow \quad Z_1 = \sqrt{Z_o R_L}$$

Example (Quarter wave transformer, Smith chart illustration)

Determine the characteristic impedance of the quarter wave transformer required to match a $300\ \Omega$ load to a $50\ \Omega$ transmission line.

$$Z_1 = \sqrt{Z_o R_L} = \sqrt{(50)(300)} = 122.5\ \Omega$$



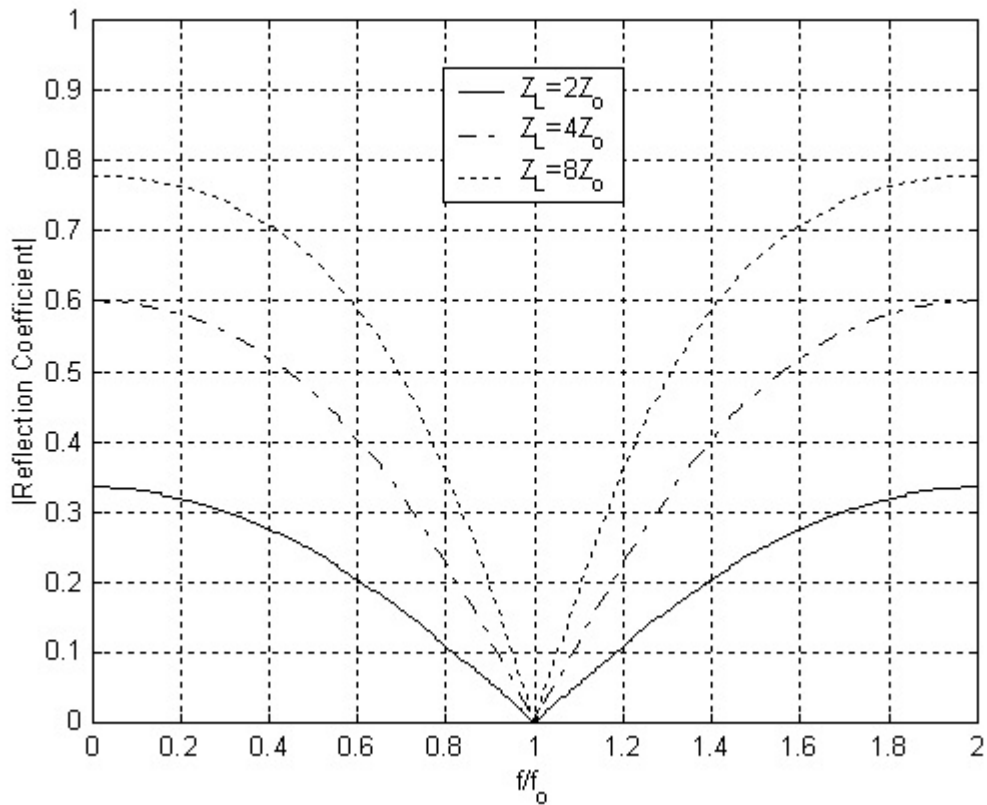
The frequency response of the quarter-wave transformer is defined according to the frequency response of the reflection coefficient:

$$\Gamma(\omega) = \frac{Z_{in}(\omega) - Z_o}{Z_{in}(\omega) + Z_o}$$

$$Z_{in}(\omega) = Z_1 \frac{R_L + jZ_1 \tan[\beta(\omega)l]}{Z_1 + jR_L \tan[\beta(\omega)l]}$$

$$Z_1 = \sqrt{Z_o R_L}$$

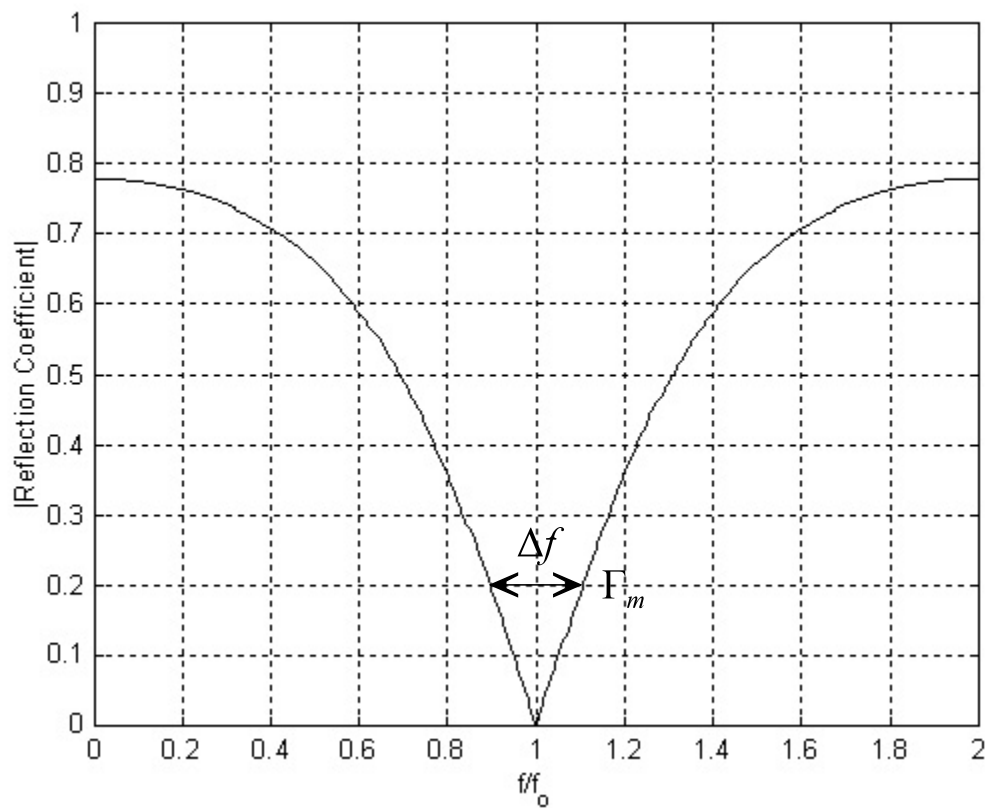
$$\beta(\omega)l = \frac{2\pi}{\lambda} \frac{\lambda_o}{4} = \frac{\pi}{2} \frac{f}{f_o}$$



The fractional bandwidth of the quarter wave transformer may be approximated by

$$\frac{\Delta f}{f_o} = 2 - \frac{4}{\pi} \cos^{-1} \left[\frac{\Gamma_m}{\sqrt{1 - \Gamma_m^2}} \frac{2\sqrt{Z_o R_L}}{|R_L - Z_o|} \right]$$

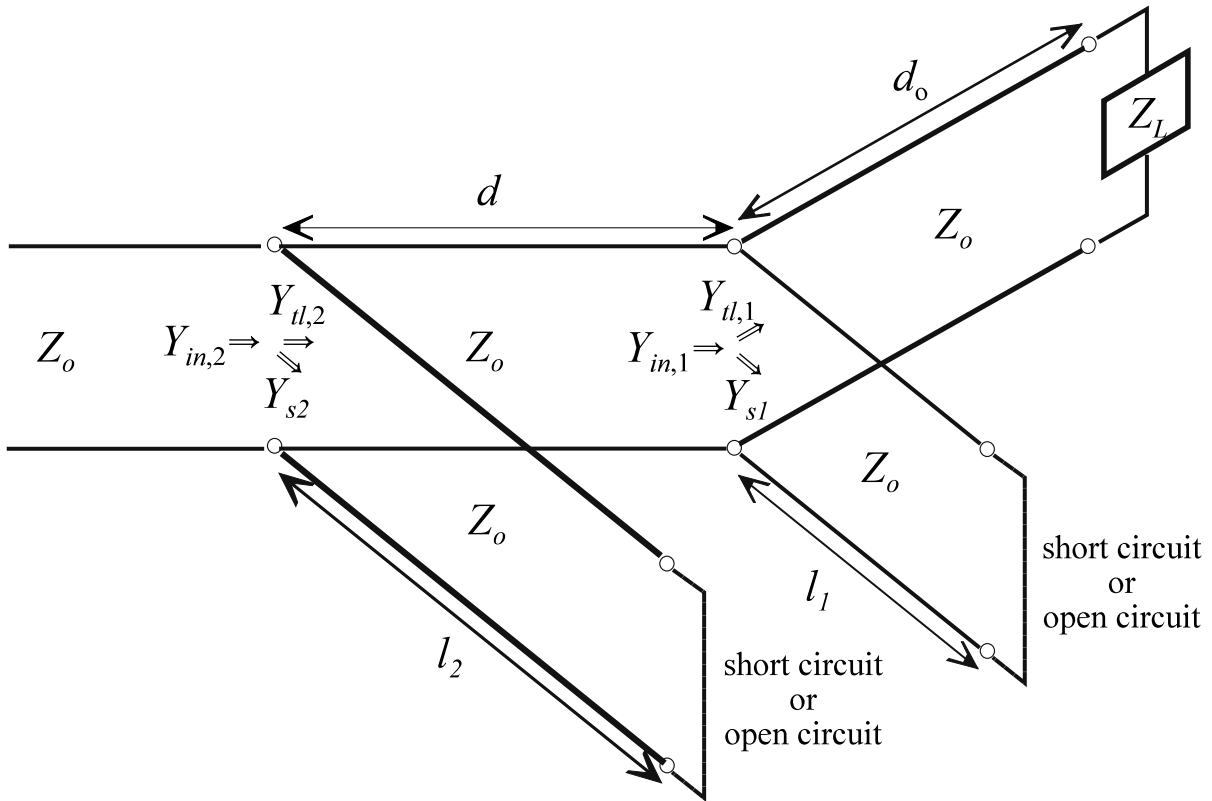
where Δf is the bandwidth for the matching network at the maximum allowable reflection coefficient magnitude Γ_m .



It should be noted that the frequency analysis of the quarter wave transformer is only valid for waveguiding structures that carry TEM waves since the characteristic impedance of guiding structures carrying non-TEM waves is frequency dependent.

Double Stub Tuner

The single stub tuner is very flexible at matching any load impedance to a given transmission line. However, if the load impedance varies so that an adjustable tuner is necessary, the single stub tuner requires that the position of the stub tuner also varies. A double stub tuner allows for an adjustable matching network that utilizes adjustable length stubs at fixed positions. The distance between the stubs is defined as d while the stub lengths are l_1 and l_2 where l_1 defines the length of the stub closest to the load.



The distance d_o between the load impedance and the first stub is somewhat arbitrary in that we may always determine the input admittance $Y_{tl,1}$ given the load impedance Z_L and the length of line d_o . However, for a given value of d_o , we may encounter some load impedances which cannot be matched for a given stub separation distance d .

The first stub must add the proper amount of susceptance to $Y_{tl,1}$ that, when transformed through the distance d to yield $Y_{tl,2}$, places this input admittance on the $g = 1$ circle. The second stub is chosen so as to cancel the susceptance of $Y_{tl,2}$.

The input admittance $Y_{in,1}$ may be written as

$$Y_{in,1} = Y_{tl,1} + Y_{s1} = G_{tl,1} + j(B_{tl,1} + B_{s1})$$

This admittance represents the equivalent load on the end of the transmission line segment of length d used to find the input admittance $Y_{tl,2}$.

$$\begin{aligned} Y_{tl,2} &= \frac{1}{Z_{tl,2}} = \frac{1}{Z_o} \frac{Z_o + jZ_{tl,1} \tan \beta d}{Z_{tl,1} + jZ_o \tan \beta d} = Y_o \frac{\frac{1}{Y_o} + j \frac{\tan \beta d}{Y_{tl,1}}}{\frac{1}{Y_{tl,1}} + j \frac{\tan \beta d}{Y_o}} \\ &= Y_o \frac{Y_{tl,1} + jY_o \tan \beta d}{Y_o + jY_{tl,1} \tan \beta d} = Y_o \frac{G_{tl,1} + j(B_{tl,1} + B_{s1} + Y_o \tan \beta d)}{Y_o + j \tan \beta d (G_{tl,1} + jB_{tl,1} + jB_{s1})} \end{aligned}$$

The real part of $Y_{in,2}$ must equal Y_o in order to match the transmission line, which yields the susceptance of the first stub. The resulting equation for B_{s1} is

$$B_{s1} = -B_{tl,1} + \frac{Y_o \pm \sqrt{(1 + \tan^2 \beta d) G_{tl,1} Y_o - G_{tl,1}^2 \tan^2 \beta d}}{\tan \beta d}$$

Note that there are two possible solutions for B_{s1} . The susceptance of the second stub must cancel that of $Y_{in,2}$. Setting B_{s1} equal to the negative of the imaginary part of $Y_{in,2}$ yields

$$B_{s2} = \frac{\pm Y_o \sqrt{Y_o G_{tl,1} (1 + \tan^2 \beta d) - G_{tl,1}^2 \tan^2 \beta d} + G_{tl,1} Y_o}{G_{tl,1} \tan \beta d}$$

The upper and lower signs on the stub susceptance equations correspond to the given pair of susceptance solutions. The lengths of the stubs are found using the susceptance equations for an open-circuited or short-circuited stub.

$$Y_{in} = -jY_o \cot \beta l_s = jB \quad (\text{short circuit})$$

$$Y_{in} = jY_o \tan \beta l_o = jB \quad (\text{open circuit})$$

or

$$l_s = -\frac{1}{\beta} \tan^{-1} \left(\frac{Y_o}{B} \right) = -\frac{\lambda}{2\pi} \tan^{-1} \left(\frac{Y_o}{B} \right)$$

$$l_o = \frac{1}{\beta} \tan^{-1} \left(\frac{B}{Y_o} \right) = \frac{\lambda}{2\pi} \tan^{-1} \left(\frac{B}{Y_o} \right)$$

Theoretically, the stub spacing d can be any value. However, stub spacings near 0 and $\lambda/2$ yield matching networks that are highly sensitive to frequency. In practice, stub spacings of $\lambda/8$ and $3\lambda/8$ are commonly used.

Example (Double stub tuner)

Design a double stub tuner using short circuited stubs separated by $\lambda/8$ to match a load impedance of $Z_L = (25 + j 40) \Omega$ to a 50Ω line. Assume the value of the given load impedance is that seen at the connection point of the first stub.

$$Y_{tl,1} = \frac{1}{Z_L} = \frac{1}{25 + j40} = 0.0112 - j0.0180 = G_{tl,1} + jB_{tl,1}$$

$$\beta d = \frac{2\pi}{\lambda} \frac{\lambda}{8} = \frac{\pi}{4}$$

Inserting into the equations for B_{s1} and B_{s2} yields

$$B_{s1} = 0.0200, 0.0560 \text{ S}$$

$$B_{s2} = -0.0120, 0.0520 \text{ S}$$

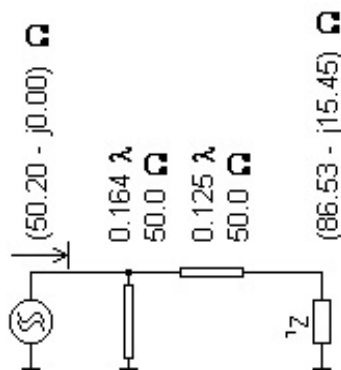
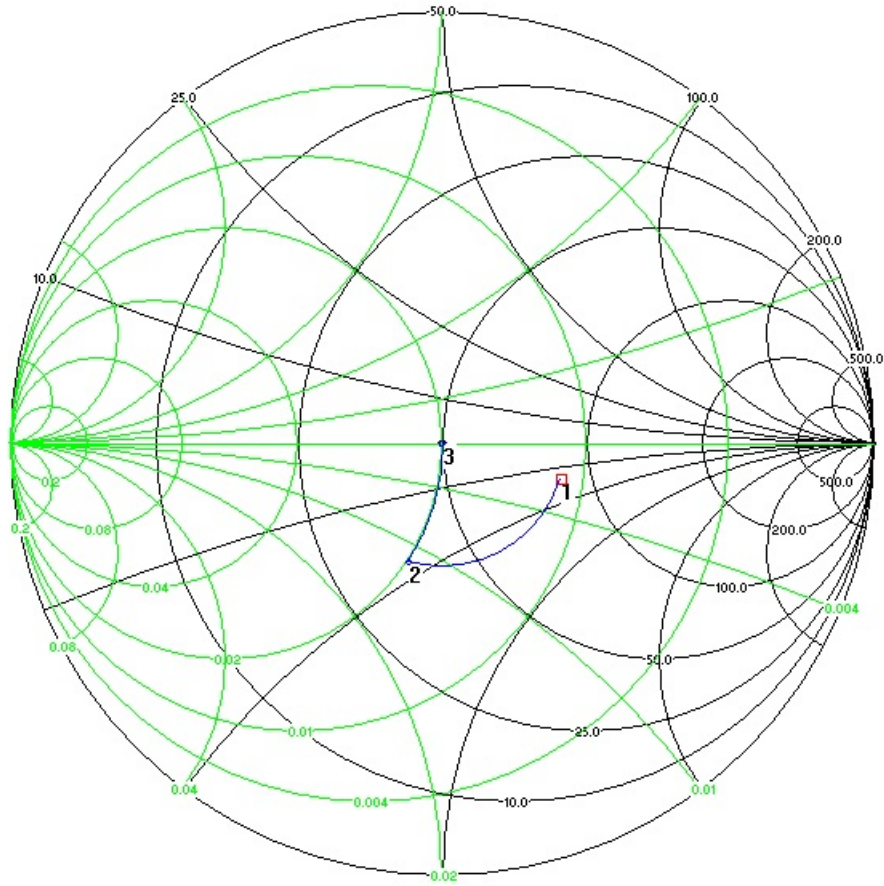
The corresponding stub lengths are

$$l_{s1} = 0.375 \lambda, 0.445 \lambda$$

$$l_{s2} = 0.164 \lambda, 0.442 \lambda$$

$$B_{s1} = 0.0200 \text{ S} \quad B_{s2} = -0.0120 \text{ S}$$

$$l_{s1} = 0.375 \lambda \quad l_{s2} = 0.164 \lambda$$



$Z_0 = 50.00 \Omega$
 $f_{\text{main}} = 500.000 \text{ MHz}$

DP #1 $(86.5 - j15.5) \Omega$
 DP #2 $(36.8 - j22.2) \Omega$
 DP #3 $(50.2 - j0.0) \Omega$

DP #1 $Z_{eq} = Z_L \parallel Z_{s1} \quad Y_{eq} = Y_L + Y_{s1}$

Frequency Response of the Double Stub Tuner

The input admittance looking into the double stub tuner ($Y_{in,2}$) is given by

$$\begin{aligned} Y_{in,2} &= Y_{tl,2} + Y_{s2} \\ &= Y_o \frac{G_{tl,1} + j(B_{tl,1} + B_{s1} + Y_o \tan \beta d)}{Y_o + j \tan \beta d (G_{tl,1} + jB_{tl,1} + jB_{s1})} + jB_{s2} \\ &= \frac{1}{Z_{in,2}} \end{aligned}$$

where the susceptances of the two stubs are

$$B_{s1} = -Y_o \cot \beta l_{s1}$$

$$B_{s2} = -Y_o \cot \beta l_{s2}$$

The corresponding reflection coefficient is

$$\Gamma(\omega) = \frac{Z_{in,2}(\omega) - Z_o}{Z_{in,2}(\omega) + Z_o}$$

In order to plot the frequency response of the reflection coefficient, we assume that the load impedance used in our double stub matching network example ($Z_L = 25 + j40 \, \Omega$ at 1 GHz) consists of a series combination of a resistor ($R = 25 \, \Omega$) and an inductor ($L = 6.37 \, \text{nH}$) [$\omega L = 40$ at $f = 1 \, \text{GHz}$]. The previously obtained solutions for the double stub matching networks are designated as double stub tuner #1 and double stub tuner #2.

Double stub tuner #1

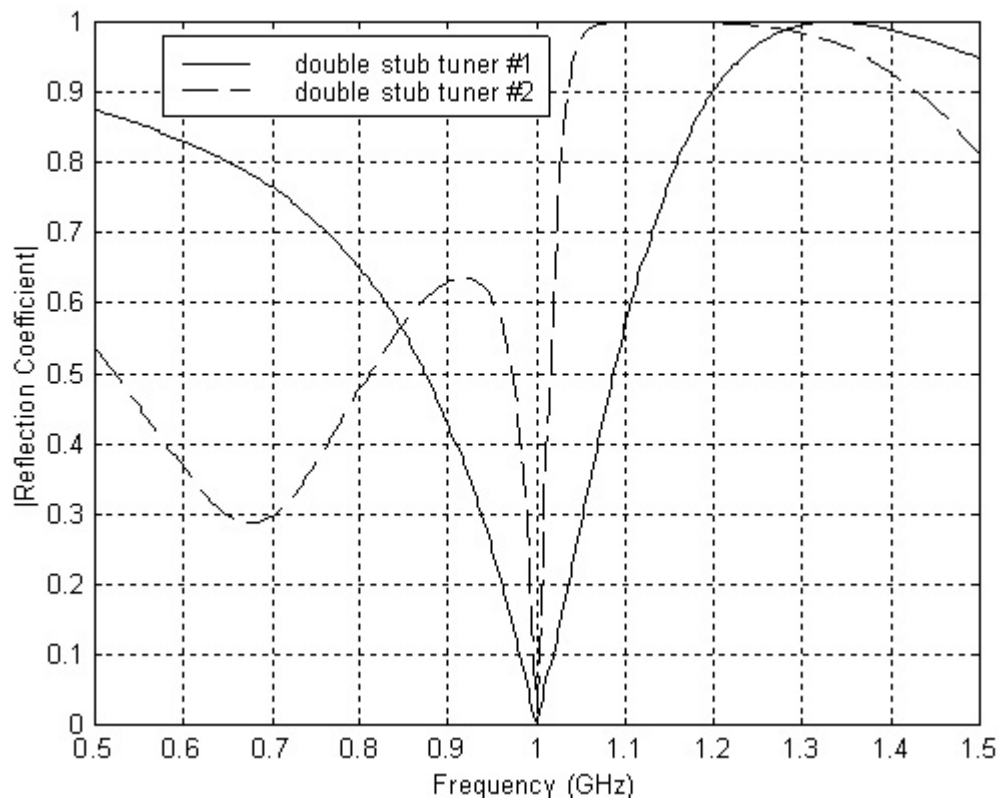
$$B_{s1} = 0.0200 \, \text{S} \, (l_{s1} = 0.375 \, \lambda)$$

$$B_{s2} = -0.0120 \, \text{S} \, (l_{s2} = 0.164 \, \lambda)$$

Double stub tuner #2

$$B_{s1} = 0.0560 \, \text{S} \, (l_{s1} = 0.445 \, \lambda)$$

$$B_{s2} = 0.0520 \, \text{S} \, (l_{s2} = 0.442 \, \lambda)$$

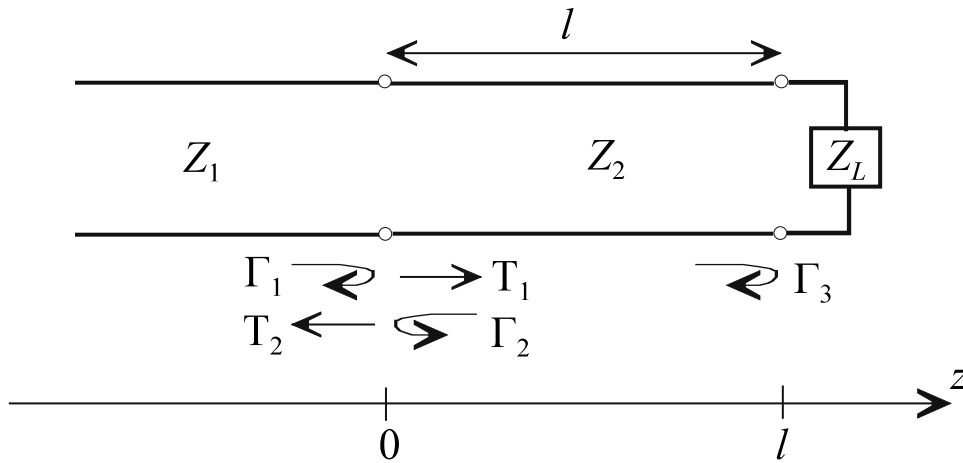


Note that the frequency response of double stub tuner #1 is much less frequency sensitive than that of double stub tuner #2. This is due to the fact that the stub lengths for tuner #2 are close to $\lambda/2$ while those of tuner #1 are not. Both designs are useful depending on the application. Double stub tuner #1 is preferable when a wider bandwidth is required. However, double stub tuner #2 may be useful in a narrowband application where the matching network performs the function of a bandpass filter.

The Theory of Small Reflections

One way of designing broadband matching networks is to use multiple sections of transmission line rather than just one section as in the case of the quarter wave transformer. In order to simplify the analysis of these multiple section matching networks, the *theory of small reflections* is utilized.

Consider a load impedance Z_L connected to a transmission line of characteristic impedance Z_1 through a section of transmission line of characteristic impedance Z_2 as shown below. The length of the connecting transmission line is l . We can easily write equations for the local (or partial) reflection and transmission coefficients at the two points on the transmission line network where reflections may occur.



$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$T_1 = 1 + \Gamma_1 = \frac{2Z_2}{Z_1 + Z_2}$$

$$\Gamma_2 = \frac{Z_1 - Z_2}{Z_1 + Z_2} = -\Gamma_1$$

$$T_2 = 1 + \Gamma_2 = \frac{2Z_1}{Z_1 + Z_2}$$

$$\Gamma_3 = \frac{Z_L - Z_2}{Z_L + Z_2}$$

We assume a wave of unit amplitude is traveling toward the load on the Z_1 section of transmission line ($e^{-j\beta z}$). Using the connection between the two transmission lines as reference ($z = 0$), the wave amplitude incident on the transmission line connection point is $1 \angle 0^\circ$. The amplitude of the reflected wave is Γ_1 while the amplitude of the transmitted wave is T_1 . The transmitted wave travels a distance l to the load connection (with a total phase shift of $e^{-j\beta l} = e^{-j\theta}$). The reflected wave at this point is

$$T_1 \Gamma_3 e^{-j\theta}$$

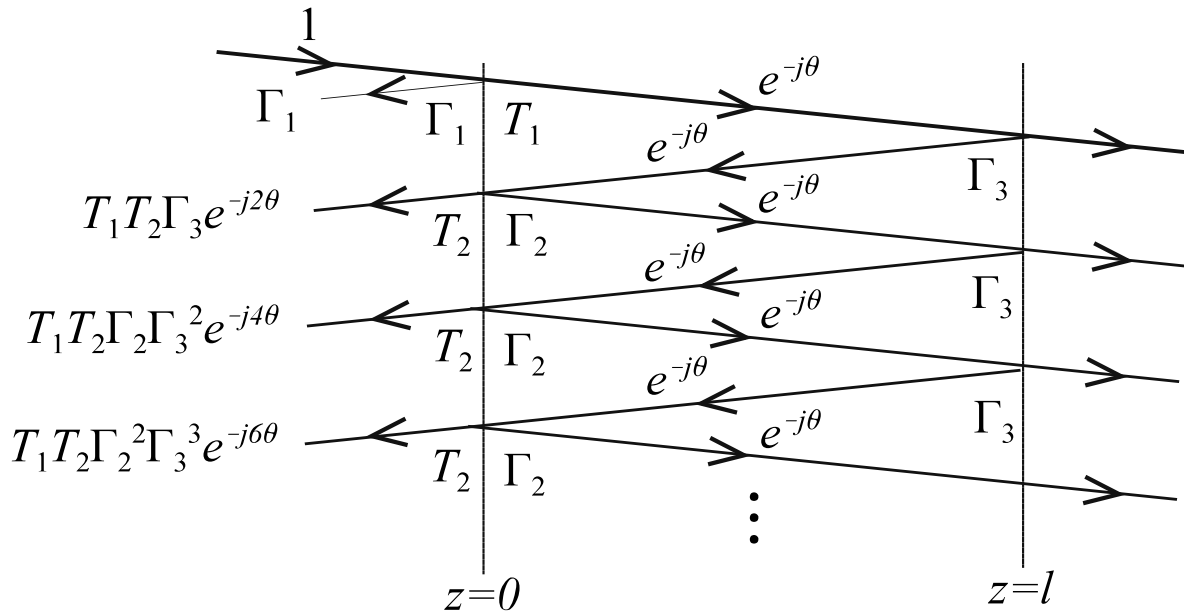
This reflected wave now travels a distance l in the opposite direction to the transmission line connection. The wave reflected back to the load at this point is (accounting for the additional $e^{-j\theta}$ phase shift)

$$T_1 \Gamma_2 \Gamma_3 e^{-j2\theta}$$

while the wave transmitted to the Z_1 transmission line is

$$T_1 T_2 \Gamma_3 e^{-j2\theta}$$

This process is then repeated for the multiple reflections at each end of the connecting transmission line section which leads to an infinite sum of reflected and transmitted signals to describe the total reflected wave.



The total reflected wave at the transmission line connection is

$$\begin{aligned}\Gamma &= \Gamma_1 + T_1 T_2 \Gamma_3 e^{-j2\theta} + T_1 T_2 \Gamma_2 \Gamma_3^2 e^{-j4\theta} + T_1 T_2 \Gamma_2^2 \Gamma_3^3 e^{-j6\theta} + \dots \\ &= \Gamma_1 + T_1 T_2 \Gamma_3 e^{-j2\theta} \sum_{n=0}^{\infty} \Gamma_2^n \Gamma_3^n e^{-j2n\theta}\end{aligned}$$

The summation in the above equation can be written in closed form using the following geometric series.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

This gives

$$\sum_{n=0}^{\infty} [\Gamma_2 \Gamma_3 e^{-j2\theta}]^n = \frac{1}{1 - \Gamma_2 \Gamma_3 e^{-j2\theta}}$$

so that the total reflected wave may be written as

$$\Gamma = \Gamma_1 + \frac{T_1 T_2 \Gamma_3 e^{-j2\theta}}{1 - \Gamma_2 \Gamma_3 e^{-j2\theta}}$$

This expression can be further simplified by noting that

$$\Gamma_2 = -\Gamma_1$$

$$T_1 = 1 + \Gamma_1$$

$$T_2 = 1 + \Gamma_2$$

which yields

$$\Gamma = \Gamma_1 + \frac{(1 - \Gamma_1^2) \Gamma_3 e^{-j2\theta}}{1 + \Gamma_1 \Gamma_3 e^{-j2\theta}} = \frac{\Gamma_1 + \Gamma_3 e^{-j2\theta}}{1 + \Gamma_1 \Gamma_3 e^{-j2\theta}}$$

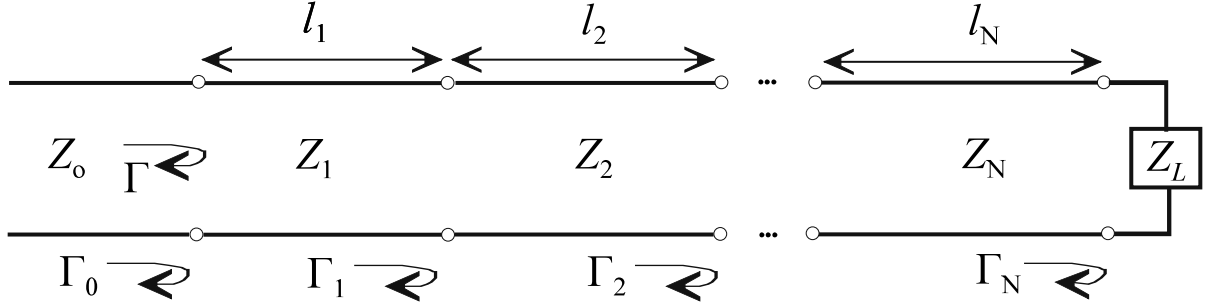
If $|\Gamma_1 \Gamma_3| \ll 1$, then the expression for the total reflection coefficient may be approximated by

$$\Gamma \approx \Gamma_1 + \Gamma_3 e^{-j2\theta}$$

This expression for the total reflection coefficient shows that the reflected wave is dominated by the initial reflection at the transmission line connection and the first reflection from the load connection.

Multisection Matching Transformer

A multisection matching transformer may be formed by connecting N transmission line sections in series between the feeder transmission line of characteristic impedance Z_o and the load impedance of Z_L .



We may define local reflection coefficients $\Gamma_0, \Gamma_1, \Gamma_2, \dots, \Gamma_N$ at the connection points of the multisection matching transformer.

$$\Gamma_0 = \frac{Z_1 - Z_o}{Z_1 + Z_o}$$

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \quad (1 \leq n \leq N-1)$$

$$\Gamma_N = \frac{Z_L - Z_N}{Z_L + Z_N}$$

If we assume that each transmission line section is of equal length, the local reflection coefficients are of the same sign (Z_n increases or decreases monotonically along the length of the transformer), and the change in impedance from section to section makes the theory of small reflections applicable, then the overall reflection coefficient of the system may be written as

$$\Gamma(\theta) \approx \Gamma_o + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \dots + \Gamma_N e^{-j2N\theta}$$

If we further assume that the reflection coefficients along the transformer are symmetric (i.e., $\Gamma_o = \Gamma_N$, $\Gamma_1 = \Gamma_{N-1}$, etc.), then

$$\Gamma(\theta) \approx e^{-jN\theta} [\Gamma_o (e^{jN\theta} + e^{-jN\theta}) + \Gamma_1 (e^{j(N-2)\theta} + e^{-j(N-2)\theta}) + \dots$$

The last term in the reflection coefficient equation is

$$\Gamma_{(N-1)/2} (e^{j\theta} + e^{-j\theta})$$

if N is odd and

$$\Gamma_{N/2}$$

if N is even. The sums of complex exponentials in the reflection coefficient equation may be written as cosines to yield

$$\Gamma(\theta) \approx 2e^{-jN\theta} \left[\Gamma_o \cos N\theta + \Gamma_1 \cos(N-2)\theta + \dots + \Gamma_n \cos(N-2n)\theta + \dots + \frac{\Gamma_{N/2}}{2} \right] \quad N - \text{even}$$

$$\Gamma(\theta) \approx 2e^{-jN\theta} \left[\Gamma_o \cos N\theta + \Gamma_1 \cos(N-2)\theta + \dots + \Gamma_n \cos(N-2n)\theta + \dots + \Gamma_{(N-1)/2} \cos \theta \right] \quad N - \text{odd}$$

These expressions are similar in form to a Fourier series. This implies that we may synthesize any frequency response given a large enough number of sections with the proper reflection coefficients.

Binomial Multisection Matching Transformer

If we may choose the parameters of the multisection matching transformer such that

$$\Gamma(\theta) = A (1 + e^{-j2\theta})^N = A [e^{-j\theta} (e^{j\theta} + e^{-j\theta})]^N = A (2e^{-j\theta} \cos\theta)^N$$

then the reflection coefficient has the following property:

$$\frac{d^n |\Gamma(\theta)|}{d\theta^n} = 0 \quad \text{at} \quad \theta = \frac{\pi}{2} \quad \text{for } n = 1, 2, \dots, N-1$$

This property defines a *maximally flat* response for the reflection coefficient at a center frequency f_o defined by

$$\theta = \beta l = \frac{\pi}{2} \quad \Rightarrow \quad l = \frac{\lambda_o}{4}$$

where λ_o defines the wavelength of the center frequency. Thus, the length of each section of the matching transformer would be one-quarter wavelength at the designed center frequency. The characteristic impedances of the individual transformer sections must be determined such that the maximally flat response is obtained.

The general form of the reflection coefficient approximation for the N -section matching transformer can easily be written in terms of a binomial series according to

$$\begin{aligned} \Gamma(\theta) &= \Gamma_o + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \dots + \Gamma_N e^{-j2N\theta} \\ &= A \sum_{n=0}^N C_n e^{-j2n\theta} \end{aligned}$$

where A is the general amplitude coefficient and C_n is the binomial coefficient given by

$$C_n = \frac{N!}{n! (N-n)!}$$

Equating coefficients for the binomial series and the reflection coefficient approximation yields

$$\Gamma_n = A C_n$$

The amplitude coefficient A may be determined by taking the limit of the reflection coefficient equations as the frequency decreases to zero. In that case,

$$\theta = \beta l = \frac{2\pi l}{\lambda} \Rightarrow 0$$

and each section of the matching transformer is zero length. This gives

$$\Gamma(0) = 2^N A = \frac{Z_L - Z_o}{Z_L + Z_o}$$

or

$$A = 2^{-N} \frac{Z_L - Z_o}{Z_L + Z_o}$$

The characteristic impedances of the adjacent sections of the binomial transformer are related by

$$\Gamma_n = A C_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n}$$

which yields

$$Z_{n+1} = Z_n \frac{1 + A C_n}{1 - A C_n}$$

The angle θ for the binomial multisection matching transformer is related to the frequency according to

$$\theta = \beta l = \frac{2\pi f}{v_p} \frac{v_p}{4f_o} = \frac{\pi}{2} \frac{f}{f_o} \quad \Rightarrow \quad f = \frac{2f_o}{\pi} \theta$$

where each section of the transformer is $\lambda/4$ in length. Thus, the bandwidth of the transformer may be defined in terms of the angle θ . If we define a maximum allowable reflection coefficient magnitude of Γ_m , then the bandwidth Δf can be written as

$$\Delta f = f_2 - f_1 = \frac{2f_o}{\pi} (\theta_2 - \theta_1)$$

where f_1 and f_2 define the lower and upper frequencies of the bandwidth, respectively. The angles θ_1 and θ_2 corresponding to the frequencies f_1 and f_2 are found by solving the equation for the overall reflection coefficient evaluated at the maximum allowable value.

$$|\Gamma(\theta)| = \Gamma_m = 2^N |A| |e^{-jN\theta}| \cos^N \theta$$

Solving this equation for θ yields

$$\theta_{1,2} = \cos^{-1} \left[\frac{1}{2} \left(\frac{\Gamma_m}{|A|} \right)^{\frac{1}{N}} \right]$$

The angle θ_1 associated with the lower frequency f_1 is the solution to the above equation which is less than $\theta_o = \pi/2$. Due to symmetry, the two angles θ_1 and θ_2 are equally spaced about θ_o . Thus, the bandwidth of the transformer may be written as

$$\begin{aligned}
\Delta f = f_2 - f_1 &= \frac{2f_o}{\pi}(\theta_2 - \theta_1) \\
&= \frac{2f_o}{\pi} [2(\theta_o - \theta_1) \\
&= \frac{4f_o}{\pi} \left\{ \frac{\pi}{2} - \cos^{-1} \left[\frac{1}{2} \left(\frac{\Gamma_m}{|A|} \right)^{\frac{1}{N}} \right] \right\} \\
&= f_o \left\{ 2 - \frac{4}{\pi} \cos^{-1} \left[\frac{1}{2} \left(\frac{\Gamma_m}{|A|} \right)^{\frac{1}{N}} \right] \right\}
\end{aligned}$$

In determining the individual characteristic impedances of the multisection binomial matching transformer, if one uses the previously derived formula

$$Z_{n+1} = Z_n \frac{1 + AC_n}{1 - AC_n}$$

significant errors may be introduced which build from the input of the matching section to the load. This is caused by the fact that the theory of small reflections was utilized to determine the binomial constants but an exact formulation was used to determine the amplitude constant A . A more accurate determination of the characteristic impedances results when the constant A is also determined using the theory of small reflections.

The local reflection coefficients for the connections of the individual transmission line sections are defined by

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} = \frac{\frac{Z_{n+1}}{Z_n} - 1}{\frac{Z_{n+1}}{Z_n} + 1}$$

According to the theory of small reflections, the discontinuities in the characteristic impedance at the junctions of the multisection matching transformer are small so that

$$\frac{Z_{n+1}}{Z_n} \approx 1$$

The local reflection coefficients may then be written in terms of the natural log of this quotient since

$$\ln x \approx 2 \frac{x - 1}{x + 1} \quad \text{for } x \approx 1$$

so that

$$\begin{aligned} \ln \left(\frac{Z_{n+1}}{Z_n} \right) &\approx 2 \frac{\frac{Z_{n+1}}{Z_n} - 1}{\frac{Z_{n+1}}{Z_n} + 1} \\ &= 2\Gamma_n = 2AC_n = \frac{2C_n}{2^N} \frac{Z_L - Z_o}{Z_L + Z_o} \approx \frac{C_n}{2^N} \ln \left(\frac{Z_L}{Z_o} \right) \end{aligned}$$

Solving this equation for Z_{n+1} in terms of Z_n gives

$$Z_{n+1} = Z_n \exp \left[\frac{C_n}{2^N} \ln \left(\frac{Z_L}{Z_o} \right) \right]$$

We can show that, using this formula, the impedance Z_{N+1} is exactly equal to Z_L as it should.

$$\begin{aligned}
Z_1 &= Z_o \exp \left[\frac{C_0}{2^N} \ln \left(\frac{Z_L}{Z_o} \right) \right] \\
Z_2 &= Z_1 \exp \left[\frac{C_1}{2^N} \ln \left(\frac{Z_L}{Z_o} \right) \right] \\
&= Z_o \exp \left[\frac{C_0}{2^N} \ln \left(\frac{Z_L}{Z_o} \right) \right] \exp \left[\frac{C_1}{2^N} \ln \left(\frac{Z_L}{Z_o} \right) \right] \\
&\quad \vdots \\
Z_{n+1} &= Z_n \exp \left[\frac{C_n}{2^N} \ln \left(\frac{Z_L}{Z_o} \right) \right] \\
&= Z_o \exp \left[\frac{C_0}{2^N} \ln \left(\frac{Z_L}{Z_o} \right) \right] \exp \left[\frac{C_1}{2^N} \ln \left(\frac{Z_L}{Z_o} \right) \right] \dots \exp \left[\frac{C_n}{2^N} \ln \left(\frac{Z_L}{Z_o} \right) \right] \\
Z_{n+1} &= Z_o \exp \left[\frac{\sum_{k=0}^n C_k}{2^N} \ln \left(\frac{Z_L}{Z_o} \right) \right]
\end{aligned}$$

The binomial series has the special property that

$$\sum_{k=0}^N C_k = 2^N$$

so that

$$Z_{N+1} = Z_o \exp \left[\frac{\sum_{k=0}^N C_k}{2^N} \ln \left(\frac{Z_L}{Z_o} \right) \right] = Z_o \exp \left[\ln \left(\frac{Z_L}{Z_o} \right) \right] = Z_L$$

Example (Binomial multisection matching transformer)

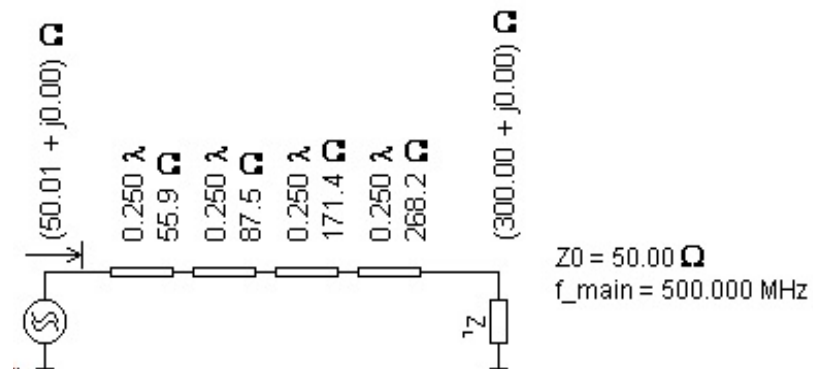
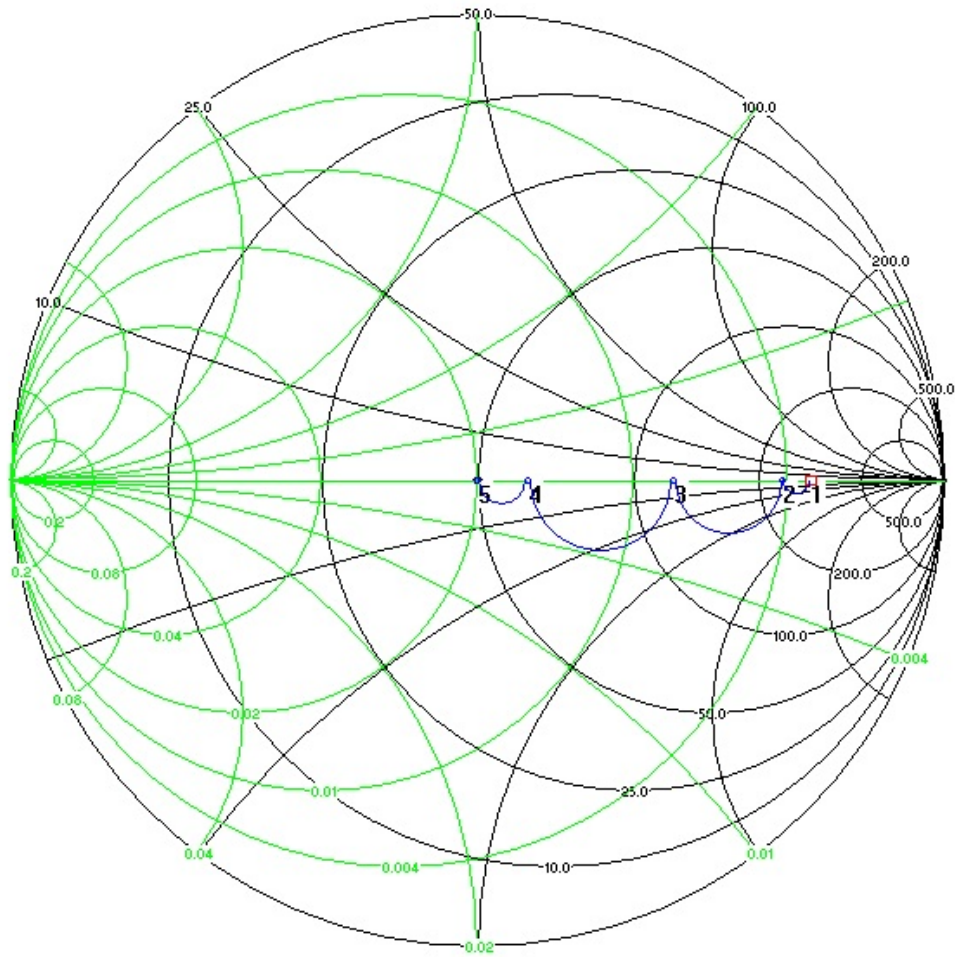
Design a 4-section binomial matching transformer to match a 300Ω load to a 50Ω line and determine the resulting percentage bandwidth for $\Gamma_m = 0.1$.

The binomial coefficients are given by

$$C_n = \frac{N!}{n!(N-n)!}$$
$$C_0 = \frac{4!}{0!4!} = 1 = C_4$$
$$C_1 = \frac{4!}{1!3!} = 4 = C_3$$
$$C_2 = \frac{4!}{2!2!} = 6$$

The impedances of the 4 transformer sections are

$$Z_{n+1} = Z_o \exp \left[\frac{\sum_{k=0}^n C_k}{2^N} \ln \left(\frac{Z_L}{Z_o} \right) \right]$$
$$Z_1 = Z_o \exp \left[\frac{C_o}{2^4} \ln \left(\frac{Z_L}{Z_o} \right) \right] = 50 \exp \left[\frac{1}{16} \ln(6) \right] = 55.9 \Omega$$
$$Z_2 = Z_o \exp \left[\frac{C_o + C_1}{2^4} \ln \left(\frac{Z_L}{Z_o} \right) \right] = 50 \exp \left[\frac{5}{16} \ln(6) \right] = 87.5 \Omega$$
$$Z_3 = Z_o \exp \left[\frac{C_o + C_1 + C_2}{2^4} \ln \left(\frac{Z_L}{Z_o} \right) \right] = 50 \exp \left[\frac{11}{16} \ln(6) \right] = 171.4 \Omega$$
$$Z_4 = Z_o \exp \left[\frac{C_o + C_1 + C_2 + C_3}{2^4} \ln \left(\frac{Z_L}{Z_o} \right) \right] = 50 \exp \left[\frac{15}{16} \ln(6) \right] = 268.2 \Omega$$



Data Point #1	$(300.0 + j0.0) \, \Omega$
Data Point #2	$(239.8 + j0.0) \, \Omega$
Data Point #3	$(122.5 + j0.0) \, \Omega$
Data Point #4	$(62.5 + j0.0) \, \Omega$
Data Point #5	$(50.0 + j0.0) \, \Omega$

The percentage bandwidth of the 4-section binomial matching network is given by

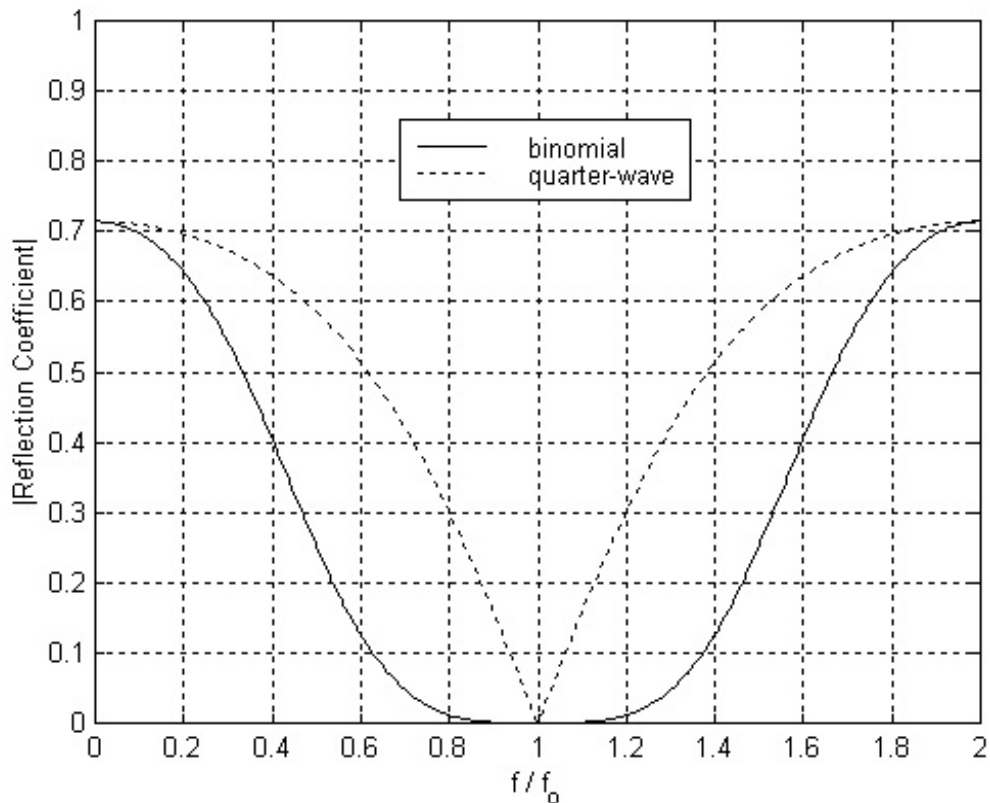
$$\frac{\Delta f}{f_o} = 2 - \frac{4}{\pi} \cos^{-1} \left[\frac{1}{2} \left(\frac{\Gamma_m}{|A|} \right)^{\frac{1}{N}} \right]$$

$$A = 2^{-N} \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{1}{16} \frac{300 - 50}{300 + 50} = \frac{5}{112}$$

$$\frac{\Delta f}{f_o} = 2 - \frac{4}{\pi} \cos^{-1} \left[\frac{1}{2} \left(\frac{(0.1)(112)}{5} \right)^{\frac{1}{4}} \right]$$

$$= 2 - \frac{4}{\pi} \cos^{-1} (0.6117)$$

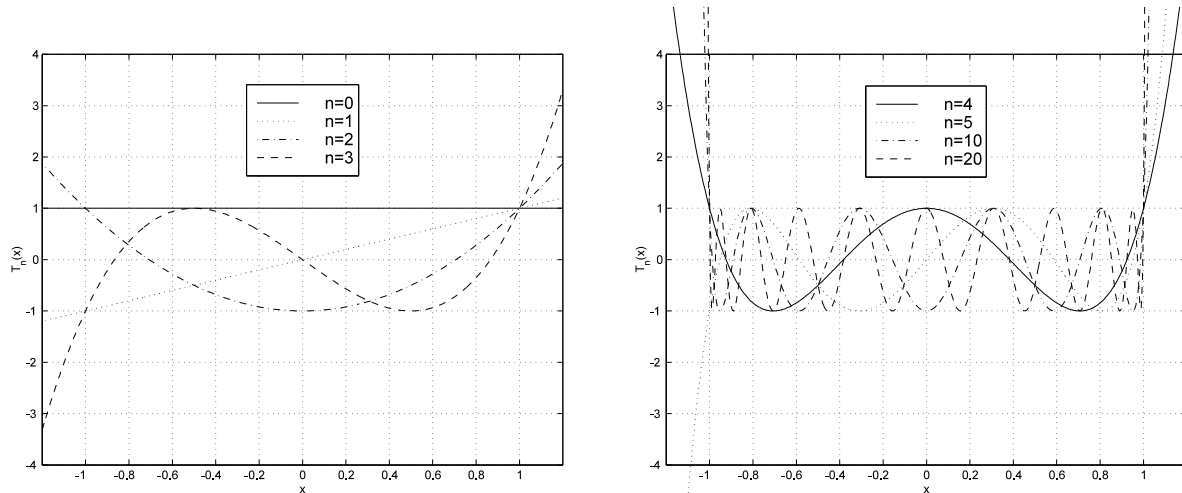
$$= 0.84 \quad (84\%)$$



Chebyshev Multisection Matching Transformer

A Chebyshev multisection matching transformer can provide even larger bandwidths than a binomial multisection matching transformer for a given number of transmission line sections. The increased bandwidth of the Chebyshev transformer comes at the cost of increased ripple over the passband of the matching network. However, we may still designate some maximum allowable reflection coefficient for the design of the Chebyshev transformer. The Chebyshev transformer exploits the characteristics of the Chebyshev polynomials.

Properties of Chebyshev Polynomials



1. Even ordered Chebyshev polynomials are even functions.
2. Odd ordered Chebyshev polynomials are odd functions.
3. The magnitude of any Chebyshev polynomial is unity or less in the range of $-1 \leq x \leq 1$.
4. $T_n(1) = 1$ for all Chebyshev polynomials.
5. All zeros (roots) of the Chebyshev polynomials lie within the range of $-1 \leq x \leq 1$.

Using the properties of Chebyshev polynomials, we may design matching networks with a reflection coefficient at or below some prescribed level over a wide bandwidth.

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \\
T_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x \\
&\vdots
\end{aligned}$$

Through the transformation of $x = \cos \theta$, the Chebyshev polynomials may be written with an argument of $\cos \theta$.

$$\begin{aligned}
T_0(\theta) &= \cos 0\theta = 1 \\
T_1(\theta) &= \cos 1\theta = \cos \theta \\
T_2(\theta) &= \cos 2\theta = 2\cos^2 \theta - 1 \\
T_3(\theta) &= \cos 3\theta = 4\cos^3 \theta - 3\cos \theta \\
T_4(\theta) &= \cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1 \\
T_5(\theta) &= \cos 5\theta = 16\cos^5 \theta - 20\cos^3 \theta + 5\cos \theta \\
T_6(\theta) &= \cos 6\theta = 32\cos^6 \theta - 48\cos^4 \theta + 18\cos^2 \theta - 1 \\
T_7(\theta) &= \cos 7\theta = 64\cos^7 \theta - 112\cos^5 \theta + 56\cos^3 \theta - 7\cos \theta \\
T_8(\theta) &= \cos 8\theta = 128\cos^8 \theta - 256\cos^6 \theta + 160\cos^4 \theta - 32\cos^2 \theta + 1 \\
T_9(\theta) &= \cos 9\theta = 256\cos^9 \theta - 576\cos^7 \theta + 432\cos^5 \theta - 120\cos^3 \theta \\
&\quad + 9\cos \theta \\
&\vdots
\end{aligned}$$

Thus, the Chebyshev polynomials may be written compactly as

$$T_n(\cos \theta) = \cos n\theta$$

The Chebyshev polynomials for all arguments x may be written as

$$\begin{aligned} T_n(x) &= \cos(n \cos^{-1} x) & \text{for } |x| < 1 \\ T_n(x) &= \cosh(n \cosh^{-1} x) & \text{for } |x| > 1 \end{aligned}$$

In order to implement the Chebyshev matching transformer, the endpoints of the required passband $(\theta_m, \pi - \theta_m)$ [with a center frequency at $\pi/2$] must be mapped onto the range where the Chebyshev polynomials satisfy

$$|T_n(\cos \theta)| \leq 1$$

This mapping is defined by

$$T_n\left(\frac{\cos \theta}{\cos \theta_m}\right) = T_n(\sec \theta_m \cos \theta)$$

The resulting Chebyshev polynomials for the given mapping are

$$\begin{aligned} T_0(\sec \theta_m \cos \theta) &= 1 \\ T_1(\sec \theta_m \cos \theta) &= \sec \theta_m \cos \theta \\ T_2(\sec \theta_m \cos \theta) &= \sec^2 \theta_m (1 + \cos 2\theta) - 1 \\ T_3(\sec \theta_m \cos \theta) &= \sec^3 \theta_m (\cos 3\theta + 3 \cos \theta) - 3 \sec \theta_m \cos \theta \\ T_4(\sec \theta_m \cos \theta) &= \sec^4 \theta_m (\cos 4\theta + 4 \cos 2\theta + 3) \\ &\quad - 4 \sec^2 \theta_m (\cos 2\theta + 1) + 1 \\ &\vdots \end{aligned}$$

We have previously shown using the theory of small reflections that the reflection coefficient of an N section multisection matching transformer may be written as

$$\begin{aligned} \Gamma(\theta) &= 2e^{-jN\theta} \left[\Gamma_o \cos N\theta + \Gamma_1 \cos(N-2)\theta + \right. \\ &\quad \left. \dots + \Gamma_n \cos(N-2n)\theta + \dots \right] \\ &= Ae^{-jN\theta} T_N(\sec \theta_m \cos \theta) \end{aligned}$$

Given that the maximum magnitude of each Chebyshev polynomial should be unity within the passband, the maximum magnitude of the reflection coefficient in the passband is

$$\Gamma_m = |A|$$

The constant A is determined by taking the limit as θ approaches 0.

$$\Gamma(0) = \frac{Z_L - Z_o}{Z_L + Z_o} = A T_N(\sec \theta_m)$$

$$A = \frac{Z_L - Z_o}{Z_L + Z_o} \frac{1}{T_N(\sec \theta_m)} \approx \frac{\ln(Z_L/Z_o)}{2 T_N(\sec \theta_m)}$$

The reflection coefficient then becomes

$$\Gamma(\theta) = \frac{1}{2} e^{-jN\theta} \ln \frac{Z_L}{Z_o} \frac{T_N(\sec \theta_m \cos \theta)}{T_N(\sec \theta_m)}$$

In order to determine the angle θ_m , we solve the following equation:

$$T_N(\sec \theta_m) = \frac{\ln(Z_L/Z_o)}{2\Gamma_m} = \cosh \left[N \cosh^{-1}(\sec \theta_m) \right]$$

$$\sec \theta_m = \cosh \left[\frac{1}{N} \cosh^{-1} \left| \frac{\ln(Z_L/Z_o)}{2\Gamma_m} \right| \right]$$

The characteristic impedances are found after the local reflection coefficients are determined.

$$\Gamma_n \approx \frac{1}{2} \ln \frac{Z_{n+1}}{Z_n} \quad \Rightarrow \quad Z_{n+1} = Z_n e^{2\Gamma_n}$$

The resulting fractional bandwidth for the Chebyshev multisection transformer is

$$\frac{\Delta f}{f_o} = 2 - \frac{4\theta_m}{\pi}$$

Example (Chebyshev multisection matching transformer)

Design a 4-section Chebyshev matching transformer to match a 300Ω load to a 50Ω line and for $\Gamma_m = 0.1$.

The equation for the reflection coefficient with $N = 4$ gives

$$\begin{aligned}\Gamma(\theta) &= 2e^{-j4\theta} [\Gamma_o \cos 4\theta + \Gamma_1 \cos 2\theta + \Gamma_2/2] \\ &= Ae^{-j4\theta} T_4(\sec \theta_m \cos \theta)\end{aligned}$$

The angle θ_m is found by

$$\sec \theta_m = \cosh \left[\frac{1}{N} \cosh^{-1} \left| \frac{\ln(Z_L/Z_o)}{2\Gamma_m} \right| \right] = \cosh \left[\frac{1}{4} \cosh^{-1} \left| \frac{\ln(6)}{2(0.1)} \right| \right]$$

which yields $\theta_m = 38.2^\circ$. Inserting the fourth order Chebyshev polynomial given by

$$\begin{aligned}T_4(\sec \theta_m \cos \theta) &= \sec^4 \theta_m (\cos 4\theta + 4 \cos 2\theta + 3) \\ &\quad - 4 \sec^2 \theta_m (\cos 2\theta + 1) + 1\end{aligned}$$

into the reflection coefficient formula gives

$$\begin{aligned}\Gamma(\theta) &= 2e^{-j4\theta} \left[\Gamma_o \cos 4\theta + \Gamma_1 \cos 2\theta + \Gamma_2/2 \right] \\ &= Ae^{-j4\theta} \left[\sec^4 \theta_m (\cos 4\theta + 4 \cos 2\theta + 3) \right. \\ &\quad \left. - 4 \sec^2 \theta_m (\cos 2\theta + 1) + 1 \right]\end{aligned}$$

Equating the coefficients associated with the cosine terms gives

$$2\Gamma_0 = A \sec^4 \theta_m \quad \Rightarrow \quad \Gamma_0 = \frac{0.1}{2} \sec^4(38.2^\circ) = 0.131$$

$$2\Gamma_1 = 4A(\sec^4 \theta_m - \sec^2 \theta_m)$$

$$\Rightarrow \quad \Gamma_1 = \frac{4(0.1)}{2} [\sec^4(38.2^\circ) - \sec^2(38.2^\circ)] = 0.201$$

$$\Gamma_2 = A(3 \sec^4 \theta_m - 4 \sec^2 \theta_m + 1)$$

$$\Rightarrow \quad \Gamma_2 = (0.1)[3 \sec^4(38.2^\circ) - 4 \sec^2(38.2^\circ) + 1] = 0.239$$

The remaining local reflection coefficients are found by symmetry:

$$\Gamma_4 = \Gamma_0$$

$$\Gamma_3 = \Gamma_1$$

The resulting characteristic impedances are

$$Z_1 = Z_o e^{2\Gamma_0} = 65.0 \, \Omega$$

$$Z_2 = Z_1 e^{2\Gamma_1} = Z_o e^{2(\Gamma_0 + \Gamma_1)} = 97.1 \, \Omega$$

$$Z_3 = Z_2 e^{2\Gamma_2} = Z_o e^{2(\Gamma_0 + \Gamma_1 + \Gamma_2)} = 156.7 \, \Omega$$

$$Z_4 = Z_3 e^{2\Gamma_2} = Z_o e^{2(\Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3)} = 234.2 \, \Omega$$

