# Chapter 6 Microwave Resonators

Part I

1. Series and Parallel Resonant Circuits
2. Loss and $Q$ Factor of a Resonant Circuit
3. Various Waveguide Resonators
4. Coupling to a Lossy Resonator

Part II

- Time-Domain Analysis of Open Cavities

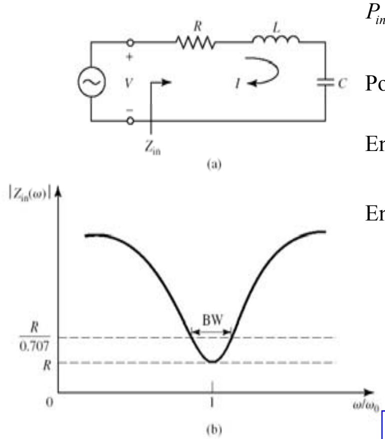
Part III

- Spectral-Domain Analysis of Open Cavities



# 6.1 Series and Parallel Resonant Circuits

(1) Series Resonant Circuit




$$Z_{in}=R+j\omega L+\left(j\omega C\right)^{-1}$$

$$P_{in}=\frac{1}{2}VI^*=\frac{1}{2}|I|^2Z_{in}=P_{loss}+2j\omega\left(W_m-W_e\right)\Rightarrow Z_{in}=\frac{2P_{in}}{|I|^2}$$

Power dissipation : $P_{\ell oss}=\frac{1}{2}|I|^2R$

Energy stored in $L$ : $W_m=\frac{1}{4}|I|^2L$

Energy stored in $C$ : $W_e=\frac{|I|^2}{4\omega^2C}$

- Resonance occurs at $W_m=W_e$ , $Z_{in}=R$, and $\omega=\omega_0=1/\sqrt{LC}$.
- *Quality factor* :

$$\boxed{Q\equiv\omega\frac{\text{Average energy stored}}{\text{Energy loss per second}}=\omega_0\frac{2W_m}{P_{loss}}=\frac{\omega_0L}{R}=\frac{1}{\omega_0RC}}$$



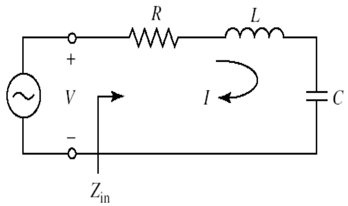
# Series Resonant Circuit

Near Resonance $\omega=\omega_0+\Delta\omega$, $\Delta\omega$ is small.

$$Z_{in}=R+j\omega L\left(1-\frac{1}{\omega^2LC}\right)$$

$$=R+j\omega L\left(\frac{\omega^2-\omega_0^2}{\omega^2}\right)$$

$$\approx R+j2\Delta\omega L=R+j2\frac{QR}{\omega_0}\Delta\omega,\ Q\equiv\frac{\omega_0L}{R}$$




Complex frequency : $\omega_0\leftarrow\omega_0\left(1+\frac{j}{2Q}\right)$

- If $R$=0 or lossless case,

$$Z_{in}\approx j2L\Delta\omega=j2L\left(\omega-\omega_0\right)$$

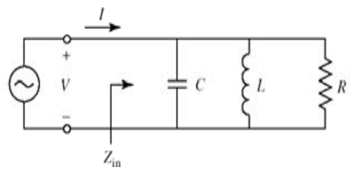
$$\rightarrow j2L\left[\omega-\omega_0\left(1+\frac{j}{2Q}\right)\right]=j2L\left(\omega-\omega_0\right)+\frac{\omega_0L}{Q}$$

$$=j2L\left(\omega-\omega_0\right)+R$$

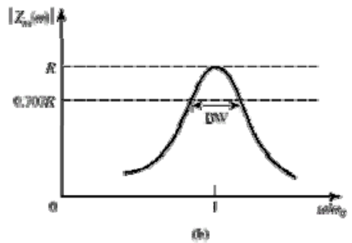
- A resonator with loss can be treated as a *lossless resonator* whose resonant frequency $\omega_0$ is replaced by a complex frequency $\omega_0(1+j/2Q)$



# Parallel Resonant Circuit




$$Z_{in}=\left[\frac{1}{R}+\frac{1}{j\omega L}+j\omega C\right]^{-1}$$




Power dissipation : $P_{loss}=\frac{1}{2}\frac{|V|^2}{R}$

Energy stored in $C$ : $W_e=\frac{1}{4}C|V|^2$

Energy stored in $L$ : $W_m=\frac{1}{4}\frac{|V|^2}{\omega^2L}$

- *Complex power* delivered to the resonator

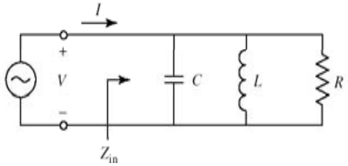
$$P_{in}=\frac{1}{2}VI^*=\frac{1}{2}\frac{|V|^2}{Z_{in}^*}=\frac{1}{2}|V|^2\left(\frac{1}{R}-\frac{1}{j\omega L}-j\omega C\right)$$

$$=P_{loss}+2j\omega\left(W_m-W_e\right)$$

- *Input impedance* at resonance,

$$Z_{in}=R,\ W_m=W_e,\ \omega_0=\frac{1}{\sqrt{LC}}$$

# Parallel Resonant Circuit




$$Q \equiv \omega \frac{\text{Average energy stored}}{\text{Energy loss per second}}$$

$$= \omega_0 \frac{2W_m}{P_{loss}} = \frac{R}{\omega_0 L} = \omega_0 RC$$

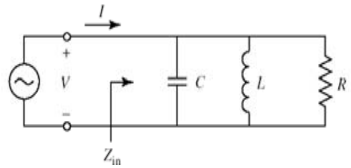
$$Z_{in} = \left[\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right]^{-1}, \text{ near resonance } \omega = \omega_0 + \Delta\omega$$

$$\approx \left[\frac{1}{R} + \frac{1 - \frac{\Delta\omega}{\omega_0}}{j\omega_0 L} + j(\omega_0 + \Delta\omega)C\right]^{-1} = \left(\frac{1}{R} + j2\Delta\omega C\right)^{-1}$$

$$= \frac{R}{1 + j2\Delta\omega RC} = \frac{R}{1 + j2Q\frac{\Delta\omega}{\omega_0}} \xrightarrow{R\to\infty} \frac{1}{j2(\Delta\omega)C}$$



# Parallel Resonant Circuit




if $R \to \infty$, $Z_{in} \approx \dfrac{1}{j2(\Delta\omega)C}$

$$2(\Delta\omega)C = j2C(\omega - \omega_0)$$

$$\omega_0 \leftarrow \omega_0(1 + j/2Q)$$

$$j2C\left[\omega - \omega_0\left(1 + \frac{j}{2Q}\right)\right]$$

$$= j2C(\omega - \omega_0) + \frac{\omega_0 C}{Q} = j2C\Delta\omega + \frac{1}{R}$$

The loss factor can be accounted by using $\omega_0 \leftarrow \omega_0(1 + j/2Q)$.

$$Q = \omega_0 \frac{2W_m}{P_{loss}} = \frac{R}{\omega_0 L} = \omega_0 RC$$

Fractional bandwidth $= 2\Delta\omega/\omega_0 = Q^{-1}$

$$|Z_{in}(\omega)| = \frac{R}{\sqrt{2}}$$

$$\frac{1}{j\omega C + (j\omega L)^{-1}} = \frac{j\omega L}{1 - \omega^2 LC} = \pm jR$$

$$\omega^2 LC - 1 = \pm \frac{\omega L}{R},$$

$\left(\dfrac{\omega}{\omega_0}\right)^2 \pm \dfrac{1}{Q}\left(\dfrac{\omega}{\omega_0}\right) - 1 = 0 \leftarrow$ Quadratic equation

$$\frac{\omega}{\omega_0} = \frac{1}{2}\left(\mp\frac{1}{Q} + \sqrt{\frac{1}{Q^2} + 4}\right) \Rightarrow \frac{\omega_1 - \omega_2}{\omega_0} = \frac{1}{Q}$$

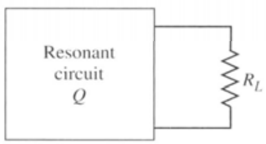


# Loaded and Unloaded $Q$

- Unloaded $Q$ (Load resistance $R_L \to \infty$)

  *Series* Resonance: $Q = (\omega_0 RC)^{-1} = \omega_0 L/R$

  *Parallel* Resonance: $Q = \omega_0 RC = R/\omega_0 L$




- External $Q$, $Q_e = \begin{cases} \dfrac{\omega_0 L}{R_L}, & \text{for series resonant circuits;} \\ \dfrac{R_L}{\omega_0 L}, & \text{for parallel resonant circuits.} \end{cases}$

- Loaded $Q$, or $Q_L$, of a resonant circuit with a load of $R_L$

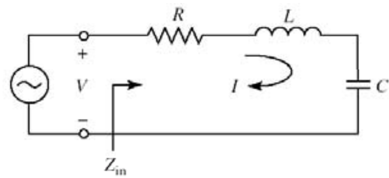
  *Series* Resonance: $Q_L = \dfrac{\omega_0 L}{R + R_L} \Rightarrow \dfrac{1}{Q_L} = \dfrac{1}{Q_e} + \dfrac{1}{Q}$

  *Parallel* Resonance: $Q_L = \dfrac{R//R_L}{\omega_0 L} \Rightarrow \dfrac{1}{Q_L} = \dfrac{1}{Q_e} + \dfrac{1}{Q}$, $\dfrac{1}{R//R_L} = \dfrac{1}{R} + \dfrac{1}{R_L}$



# Summary

(1) Model for a Series Resonant Circuit

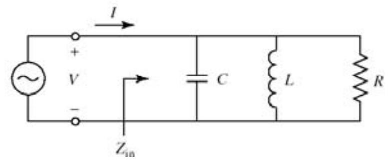



$$Q \equiv \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$

$$Z_{in} \approx R + j2L\Delta\omega = R + j2(\omega - \omega_0)L$$

- A lossless series resonant circuit is short-circuited at $\omega_0$ and has $Q \to \infty$

(2) Model for a Parallel Resonant Circuit



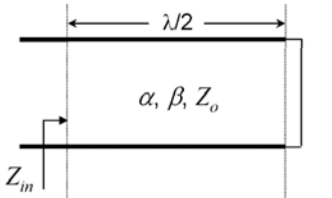

$$Q \equiv \frac{R}{\omega_0 L} = \omega_0 RC$$

$$Y_{in} \approx \frac{1}{R} + j2C\Delta\omega = \frac{1}{R} + j2C(\omega - \omega_0)$$

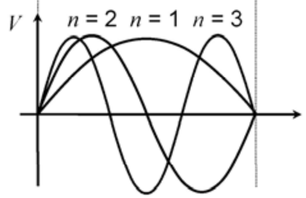
- A lossless parallel resonant circuit is short-circuited at $\omega_0$ and has $Q \to \infty$

# 6.2 Transmission Line Resonators

- All realistic resonators have a finite $Q=f_0/BW$
  - ⇨ Nonzero bandwidth ⇨ Resonator is lossy
  - ⇨ Transmission line resonator has an $\alpha \neq 0$
  - ⇨ Lossy transmission line



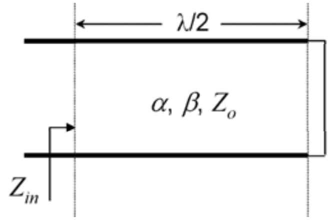

(1) Short circuit λ/2 line ⇨ series resonance
- If loss is small, $\alpha\ell \ll 1$, $\tan\alpha\ell \approx \alpha\ell$.
- Near resonance, $\omega = \omega_0 + \Delta\omega$, $\Delta\omega$ is small.
- What is the equivalent circuit?






# 6.2 Transmission Line Resonators

(1) Short-circuit λ/2 line section
(a) $\tanh\alpha\ell \approx \alpha\ell$, (b) $\Delta\omega$ is small.



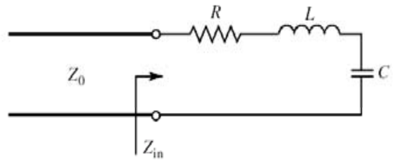

$$\beta\ell = \frac{\omega_0\ell + \Delta\omega\ell}{v_p} = \pi\left(1+\frac{\Delta\omega}{\omega_0}\right) \Rightarrow \tan\beta\ell \approx \frac{\Delta\omega}{\omega_0}\pi$$

$$Z_{in} = Z_0\tanh\left(\alpha + j\beta\right)\ell = Z_0\frac{\tanh\alpha\ell + j\tan\beta\ell}{1+j\tanh\alpha\ell\tan\beta\ell}$$

$$\approx Z_0\frac{\alpha\ell + j\frac{\Delta\omega}{\omega_0}\pi}{1+j\alpha\ell\frac{\Delta\omega}{\omega_0}\pi} \approx Z_0\left(\alpha\ell + j\frac{\Delta\omega}{\omega_0}\pi\right)$$

$$\rightarrow R + 2jL\Delta\omega$$

Equivalent circuit :



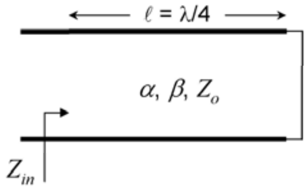

$$R = Z_0\alpha\ell,\ L = \frac{\pi}{2}\frac{Z_0}{\omega_0},\ C = \frac{1}{\omega_0^2 L}$$

$$Q = \frac{\omega_0 L}{R} = \frac{\pi}{2\alpha\ell} = \frac{\beta}{2\alpha}$$



# Transmission Line Resonators

(2) Short-circuit λ/4 line ⇨ parallel resonance
- If loss is small, $\alpha\ell \ll 1$, $\tan\alpha\ell \approx \alpha\ell$.
- Near resonance, $\omega = \omega_0 + \Delta\omega$, $\Delta\omega$ is small.




$$Z_{in} = Z_0\tanh\left(\alpha + j\beta\right)\ell = Z_0\frac{1-j\tanh\alpha\ell\cot\beta\ell}{\tanh\alpha\ell - j\cot\beta\ell}$$

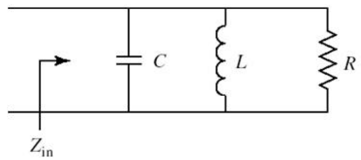
$$\beta\ell = \frac{\pi}{2}\left(1+\frac{\Delta\omega}{\omega_0}\right),\ \cot\beta\ell = -\tan\left(\frac{\pi}{2}\frac{\Delta\omega}{\omega_0}\right) \approx -\frac{\pi}{2}\frac{\Delta\omega}{\omega_0}$$

$$Z_{in} \approx \frac{Z_0}{\alpha\ell + j\frac{\pi}{2}\frac{\Delta\omega}{\omega_0}} \equiv \frac{1}{\frac{1}{R}+2j\Delta\omega C}$$

$$C = \frac{\pi}{4\omega_0 Z_0},\ R = \frac{Z_0}{\alpha\ell},\ L = \frac{1}{\omega_0^2 C}$$

$$Q = \omega_0 RC = \frac{\pi}{4\alpha\ell} = \frac{\beta}{2\alpha},\quad \ell = \frac{\pi}{2\beta} \text{ at resonance.}$$

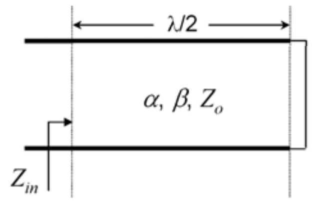
Equivalent circuit :






# Transmission Line Resonators

(3) Open-circuit λ/2 line ⇨ parallel resonance
- If loss is small, $\alpha\ell \ll 1$, $\tan\alpha\ell \approx \alpha\ell$.
- Near resonance, $\omega = \omega_0 + \Delta\omega$, $\Delta\omega$ is small.




$$Z_{in} = Z_0\coth\left(\alpha + j\beta\right)\ell = Z_0\frac{1+j\tanh\alpha\ell\tan\beta\ell}{\tanh\alpha\ell + j\tan\beta\ell}$$

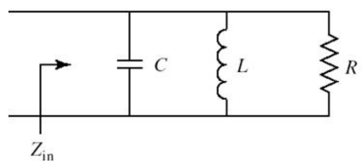
$$\beta\ell = \pi\times\left(1+\frac{\Delta\omega}{\omega_0}\right),\ \tan\beta\ell = \tan\left(\pi\frac{\Delta\omega}{\omega_0}\right) \approx \pi\frac{\Delta\omega}{\omega_0}$$

$$Z_{in} \approx \frac{Z_0}{\alpha\ell + j\pi\frac{\Delta\omega}{\omega_0}} \equiv \frac{1}{\frac{1}{R}+2j\Delta\omega C}$$

$$C = \frac{\pi}{2\omega_0 Z_0},\ R = \frac{Z_0}{\alpha\ell},\ L = \frac{1}{\omega_0^2 C}$$

$$Q = \omega_0 RC = \frac{\beta}{2\alpha},\quad \ell = \frac{\pi}{\beta} \text{ at resonance.}$$

Equivalent circuit :



# Example 6.2
## A half-wave microstrip resonator

50-Ω line, $\lambda/2$ resonator, $d = 1.59$ mm, $\varepsilon_r = 2.2$, $\tan\delta = 10^{-3}$, $f = 5$ GHz.
Calculate its $Q$ value.

Sol: $Z_0$=50 Ω, $d$=1.59 mm, and $\varepsilon_r$=2.2 $\Rightarrow$ $W$=4.9 mm, $\varepsilon_{reff}$=1.87.

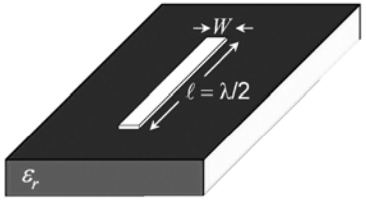
$$\ell = \lambda/2 = c\Big/\left(2f\sqrt{\varepsilon_{reff}}\right) = 21.9 \text{ mm}$$

$$\beta = 2\pi/\lambda = 143.2 \text{ rad/m}$$

$$\alpha_c = R/Z_0W = 0.075 \text{ Np/m}$$

$$\alpha_d = \ldots = 0.0611 \text{ Np/m}$$

$$Q = \beta/2\alpha = \beta/2(\alpha_c + \alpha_d) = 526.$$






# 6.3 Rectangular Waveguide Cavity

$TE_{mn}$, $TM_{mn}$ modes :

$$E_t(x,y,z) = \underline{e}(x,y)\left(A^+e^{-j\beta_{mn}z} + A^-e^{+j\beta_{mn}z}\right)$$

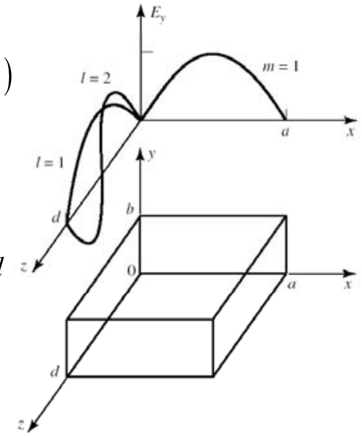
$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

For a resonant cavity, $\underline{E}_t = 0$, at $z = 0, d$

$$\beta_{mn}d = \ell\pi,\ \ell = 1,\ 2,\ 3,\ \ldots$$

$$k_{mn\ell} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2}$$

$$f_{mn\ell} = \frac{c}{2\pi}\frac{k_{mn\ell}}{\sqrt{\mu_r\varepsilon_r}}$$






# $Q$-Factor of a $TE_{10l}$ Cavity

- Total fields:

$$E_y = E_0 \sin\frac{\pi x}{a}\sin\frac{\ell\pi z}{d}$$

$$H_x = -j\frac{E_0}{Z_{TE}}\sin\frac{\pi x}{a}\cos\frac{\ell\pi z}{d}$$

$$H_z = -j\frac{j\pi E_0}{ka\eta}\cos\frac{\pi x}{a}\sin\frac{\ell\pi z}{d}$$

- Total losses = $P_c + P_d$

$$Q = \left(\frac{1}{Q_c} + \frac{1}{Q_d}\right)^{-1}$$

- $Q$ factor $\Rightarrow W_e$, $W_m$, $P_c$, and $P_d$

$$W_e = \frac{\varepsilon}{4}\iiint|\underline{E}|^2\,dv = \frac{abd}{16}\varepsilon E_0^2$$

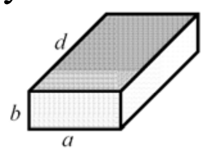
$$W_m = \frac{\varepsilon}{4}\iiint|\underline{H}|^2\,dv = \frac{abd}{16}\mu E_0^2\left(\frac{1}{Z_{TE}^2} + \frac{\pi^2}{k^2a^2\eta^2}\right) = W_e$$

$$P_c = \frac{1}{2}\iint_{walls}|\underline{H}_t|^2\,ds = \frac{R_sE_0^2\lambda^2}{8\eta^2}\left(\frac{\ell^2ab}{d^2} + \frac{bd}{a^2} + \frac{\ell^2a}{2d} + \frac{d}{2a}\right)$$

$$Q_c = \omega_0\frac{2W_e}{P_c} = \frac{(kad)^3 b\eta}{2\pi^2R_s}\frac{1}{2\ell^2a^3b + 2bd^3 + \ell^2a^3d + ad^3}$$

$$P_d = \frac{\omega\varepsilon''}{2}\iiint_v|E|^2\,dv = \frac{\omega\varepsilon''}{2}\frac{abd}{4}E_0^2$$

$$Q_d = \omega_0\frac{2W_e}{P_d} = \frac{\varepsilon''}{\varepsilon'} = (\tan\delta)^{-1}$$






# Example 6.3
## Design of a Rectangular Waveguide Cavity

$a = 40$ mm, $b = 20.0$ mm, $\varepsilon_r = 2.25$, $\tan\delta = 4\times10^{-4}$. If $f = 5$ GHz, find $d$ and $Q$ values for $\ell = 1$.

Sol:

$$k = \frac{\omega}{c}\sqrt{\varepsilon_r} = \frac{2\pi\times5\times10^9}{3\times10^{10}}\sqrt{2.25} = 1.5708 \text{ cm}^{-1}$$

$$\beta d = \ell\pi \Rightarrow d = \frac{\ell\pi}{\sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}} = 23.05 \text{ mm, for } \ell = 1.$$

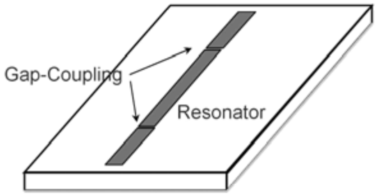
$$R_s = 1.84\times10^{-2}\ \Omega(\text{copper@5GHz}),\ \eta = \frac{120\pi}{\sqrt{\varepsilon_r}} = 251.3\ \Omega$$

$$Q_d = (\tan\delta)^{-1} = 2500,\ Q_c = 8370,\ \ell = 1.$$

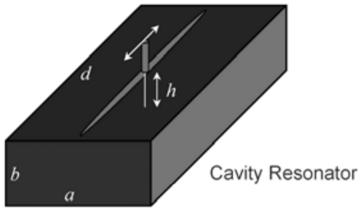
$$Q = \left(\frac{1}{Q_c} + \frac{1}{Q_d}\right)^{-1} = \left(\frac{1}{8370} + \frac{1}{2500}\right)^{-1} = 1925,\ \ell = 1.$$

# 6.6 Excitation of Resonators

Gap-Coupling to a
Micro resonator cavity



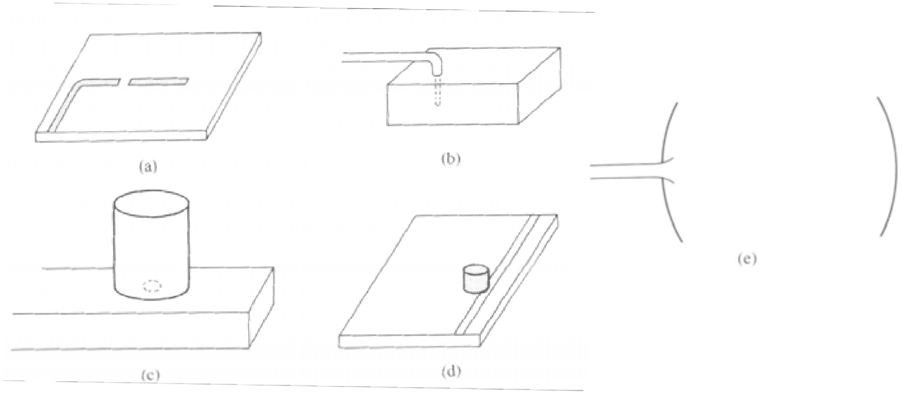

Probe-coupling to
A rectangular WG cavity




(1) The penetration depth $h$ is tunable
For impedance adjustment.
(2) The probe can be sliding along $z$.



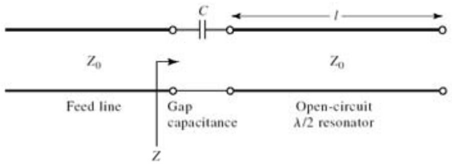
# Coupling to Microwave Resonators

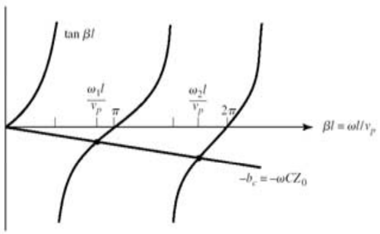



(a) A microstrip transmission line resonator gap coupled to a microstrip feed line.
(b) A rectangular cavity resonator fed by a coaxial probe.
(c) A circular cavity resonator aperture coupled to a rectangular waveguide.
(d) A dielectric resonator coupled to a microstrip feed line.
(e) A Fabry-Perot resonator fed by a waveguide horn antenna.



# A Gap-Coupled Microstrip Resonator




- Temporarily treat the lossy resonator as losses and apply the concept of *complex frequency* to evaluate its loss term $R$

$$b_c = Z_0\omega C; \qquad z_{in} = \frac{Z_{in}}{Z_0} = -j\frac{\frac{1}{\omega C} + Z_0\cot\beta\ell}{Z_0} = -j\frac{b_c + \tan\beta\ell}{b_c\tan\beta\ell}$$

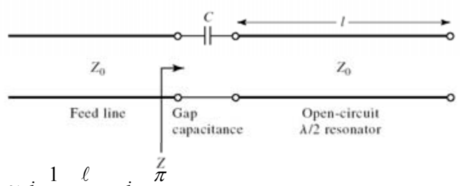
- Resonance occurs at $z_{in} = 0,\ b_c + \tan\beta\ell = 0,\ \tan\beta\ell = -b_c$ ,
- If $b_c \ll 1,\ \omega_1$ is close to the first resonant frequency of the unloaded resonator.
- The coupling of the capacitor C will lower its resonant frequency.



# The Coupling Coefficient $g$

$$b_c = Z_0\omega C$$

$$z_{in} = \frac{Z_{in}}{Z_0} = -j\frac{\frac{1}{\omega C} + Z_0\cot\beta\ell}{Z_0} = -j\frac{b_c + \tan\beta\ell}{b_c\tan\beta\ell}$$



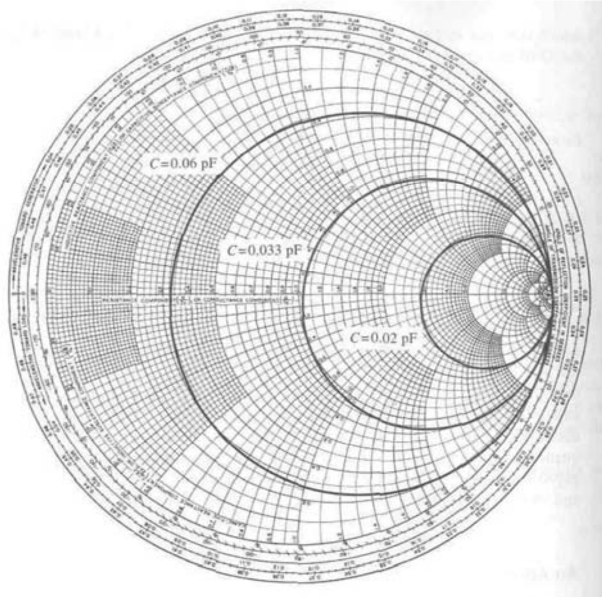

$$\frac{dz_{in}(\omega)}{d\omega} = -j\frac{\sec^2\beta\ell}{b_c\tan\beta\ell}\frac{d(\beta\ell)}{d\omega} = j\frac{\left(1+b_c^2\right)}{b_c^2}\frac{\ell}{v_p} \approx j\frac{1}{b_c^2}\frac{\ell}{v_p} = j\frac{\pi}{\omega_1 b_c^2}$$

$$z_{in}(\omega) = z_{in}(\omega_1) + (\omega-\omega_1)\left.\frac{dz_{in}(\omega)}{d\omega}\right|_{\omega_1} + \cdots = (\omega-\omega_1)\frac{j\pi}{\omega_1 b_c^2} \xrightarrow{\omega_1 \leftarrow \omega_1\left(1+j\frac{1}{2Q}\right)} \frac{\pi}{2Qb_c^2} + \frac{j\pi(\omega-\omega_1)}{\omega_1 b_c^2}$$

The input resistance $R = \dfrac{Z_0\pi}{2Qb_c^2}$ and coupling cofficient $g = \dfrac{Z_0}{R} = \dfrac{2Qb_c^2}{\pi}$.

$b_c < \sqrt{\pi/2Q},\ g < 1,$ undercoupled
$b_c = \sqrt{\pi/2Q},\ g = 1,$ critically coupled
$b_c > \sqrt{\pi/2Q},\ g > 1,$ overcoupled

# Smith chart for the gap-coupled microstrip resonator






# Example 6.6
# Design of Gap-Coupled Microstrip Resonator

50-Ω microstrip feedline, 50-Ω microstrip $\lambda/2$ resonator with $\ell = 21.75$ mm, $\varepsilon_{reff} = 1.9$, $\alpha = 0.001$ dB/mm. Find the coupling capacitor and the resonant frequency $f_1$.

Sol:

$$f_0 = \frac{v_p}{\lambda_g} = \frac{c}{2\ell\sqrt{\varepsilon_{reff}}} = \frac{3\times10^{11}}{2\times21.75\times\sqrt{1.9}} = 5 \text{ GHz}$$

$$Q = \frac{\beta}{2\alpha} = \frac{\pi}{2\ell\alpha} = 628(\text{or } 200\times\pi)$$

$$b_c = \sqrt{\frac{\pi}{2Q}} = \sqrt{\frac{\pi}{2\times200\pi}} = 0.05$$

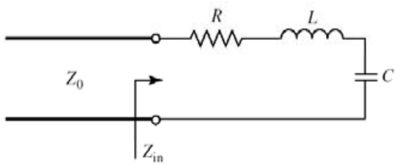
$$C = \frac{b_c}{\omega Z_0} = \frac{0.05}{2\pi\times5\times10^9\times50} = 0.032 \text{ pF}$$

$$f_1 = 4.918 \text{ GHz(obtained by a root-searching process)}$$



# 6.6 Excitation of Resonators

*Critical coupling* : A resonator is matched to a feedline to have *max power* transfer at *resonant frequency*.




$Z_{in} = R + 2jL\Delta\omega$

Max power transfer → Conjugate matching

$$\left.\frac{\partial P}{\partial R_{in}}\right| = \left.\frac{\partial P}{\partial X_{in}}\right| = 0, \text{ See Chapter 2.}$$

$$\Rightarrow Z_{in} = Z_g^*,\ R_{in} = R_g\ ,\ X_{in} = -X_g$$

- At resonance, $\omega = \omega_0$, or $\Delta\omega = 0$, $Z_{in} = R = Z_0$
  Unloaded $Q = \omega_0 L/R$
  External $Q_e = \omega_0 L/Z_0$

- Coupling coefficient $g = Q/Q_e = Z_0/R$
  $g < 1$, resonator is undercoupled to the feedline.
  $g = 1$, resonator is critically coupled to the feedline.
  $g > 1$, resonator is overcoupled to the feedline.



# Smith chart illustrating coupling to series *RLC* circuit