3 Matrices and Determinants

3.1 Introduction

While solving linear systems of equations, a new notation was introdu[ced to] reduce the amount of writing. For this new notation the word *matrix* was first use[d by] the English mathematician James Sylvester (1814 – 1897). Arthur C[ayley] (1821 – 1895) developed the theory of matrices and used them in the li[near] transformations. Now-a-days, matrices are used in high speed computers and al[so in] other various disciplines.

The concept of determinants was used by Chinese and Japanese bu[t the] Japanese mathematician Seki Kowa (1642 – 1708) and the German Mathemat[ician] Gottfried Wilhelm Leibniz (1646 – 1716) are credited for the inventio[n of] determinants. G. Cramer (1704 – 1752) employed the determinants successfully [for] solving the systems of linear equations.

A rectangular array of numbers enclosed by a pair of brackets such as:

$$\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix} \quad \text{(i)} \qquad \text{or} \qquad \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 4 \\ 3 & 2 & 6 \\ 4 & 1 & -1 \end{bmatrix} \quad \text{(ii)}$$

is called a **matrix**. The horizontal lines of numbers are called **rows** and the ve[rtical] lines of numbers are called **columns**. The numbers used in rows or columns ar[e said] to be the **entries** or **elements** of the matrix.

The matrix in (i) has two rows and three columns while the matrix in (ii) [has four] rows and three columns. Note that the number of the elements of th[e matrix] in [(ii) is] $4 \times 3 = 12$. Now we give a general definition of a matrix.

Generally, a bracketed rectangular array of $m \times n$ elements a_{ij} $(i = 1, 2, 3, \ldots, m;$ $j = 1, 2, 3, \ldots, n)$, arranged in m rows and n columns such as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \qquad \text{(iii)}$$

is called an m by n matrix (written as $m \times n$ matrix).

$m \times n$ is called the *order* of the matrix in (iii). We usually use capital letters such as A, B, C, X, Y, etc., to represent the matrices and small letters such as a, b, c, l, m, n, $a_{11}, a_{12}, a_{13}, \ldots$, etc., to indicate the entries of the matrices.

Let the matrix in (iii) be denoted by A. The ith row and the jth column of A are indicated in the following tabular representation of A.

$$\begin{array}{rl} & \hspace{6em} j\text{th column} \\ & \hspace{8em} \downarrow \\ \begin{array}{r} A = \\ i\text{th row} \rightarrow \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array} \qquad \text{(iv)}$$

The elements of the ith row of A are a_{i1} a_{i2} a_{i3} .... a_{ij} .... a_{in} while the elements of the jth column of A are a_{1j} a_{2j} a_{3j} .... a_{ij} .... a_{mj}. We note that a_{ij} is the element of the ith row and jth column of A. The double subscripts are useful to name the elements of the matrices. For example, the element 7 is at a_{23} position in the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix}$. For convenience, we shall write the matrix A as:

$$A = [a_{ij}]_{m \times n} \text{ or } A = [a_{ij}], \text{ for } i = 1, 2, 3, \ldots, m;\ j = 1, 2, 3, \ldots, n, \text{ where}$$

a_{ij} is the element of the ith row and jth column of A.

Note: a_{ij} is also known as the (i, j)th element or entry of A.

The elements (entries) of matrices need not always be numbers but in the study of matrices, we shall take the elements of the matrices from $\mathcal{R}$ or from C.

Note: The matrix A is called real if all of its elements are real.

Row Matrix or Row vector: A matrix, which has only one row, i.e., a $1\times n$ matrix of the form $\begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \end{bmatrix}$ is said to be a row matrix or a row vector.

Column Matrix or Column Vector: A matrix which has only one column, i.e., an $m\times 1$ matrix of the form $\begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{bmatrix}$ is said to be a column matrix or a column vector.

For example $[1 \quad -1 \quad 3 \quad 4]$ is a row matrix having 4 columns and $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ is a column matrix having 3 rows.

Rectangular Matrix: If $m \neq n$, then the matrix is called a rectangular matrix of order $m\times n$, that is, the matrix in which the number of rows is not equal to the number of columns, is said to be a rectangular matrix. For example;

$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 & 0 \\ 1 & 2 & 4 \\ 3 & -1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ are rectangular matrices of orders 2×3 and 4×3 respectively.

Square Matrix: If $m = n$, then the matrix of order $m\times n$ is said to be a square matrix of order n or m. i.e., the matrix which has the same number of rows and columns called a square matrix. For example; $[0]$, $\begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 8 \\ 3 & 5 & 4 \end{bmatrix}$ are square matrices of orders 1, 2 and 3 respectively.

Let $A = [a_{ij}]$ be a square matrix of order n, then the entries $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ form the **principal diagonal** for the matrix A and the entries $a_{1n}, a_{2\,n-1}, a_{3\,n-2}, \dots, a_{n-1\,2}, a_{n1}$ form the secondary diagonal for the matrix A. For example, in the matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$, the entries of the principal diagonal are $a_{11}, a_{22}, a_{33}, a_{44}$ and the entries of the secondary diagonal are $a_{14}, a_{23}, a_{32}, a_{41}$.

The principal diagonal of a square matrix is also called the leading diagonal or main diagonal of the matrix.

Diagonal Matrix: Let $A = [a_{ij}]$ be a square matrix of order n.

If $a_{ij} = 0$ for all $i \neq j$ and at least one $a_{ij} \neq 0$ for $i = j$, that is, some elements of the principal diagonal of A may be zero but not all, then the matrix A is called a diagonal matrix. The matrices

$$[7], \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$ are diagonal matrices.

Scalar Matrix: Let $A = [a_{ij}]$ be a square matrix of order n.

If $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = k$ (some non-zero scalar) for all $i = j$, then the **matrix A is called a scalar matrix of order n. For example;**

$$\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$ are scalar matrices of order 2, 3

respectively.

Unit Matrix or Identity Matrix: Let $A=[a_{ij}]$ be a square matrix of order n. If a_{ij} for all $i \neq j$ and $a_{ij} = 1$ for all $i = j$, then the matrix A is called a *unit matrix* or *identity matrix* of order n. We denote such a matrix by I_n and it is of the form:

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

The identity matrix of order 3 is denoted by I_3, that is, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Null Matrix or Zero Matrix: A square or rectangular matrix whose each element zero, is called a *null or zero matrix*. An $m \times n$ matrix with all its elements equal zero, is denoted by $O_{m\times n}$. Null matrices may be of any order. Here are examples:

$$[0], [0 \quad 0 \quad 0], \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

O may be used to denote null matrix of any order if there is no confusion.

Equal Matrices: Two matrices of the same order are said to be equal if their corresponding entries are equal. For example, $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are equal, i.e., $A = B$ iff $a_{ij} = b_{ij}$ for $i = 1,2,3,\ldots\ldots, m$, $j = 1,2,3,\ldots\ldots, n$. In other words, A and B represent the same matrix.

3.1.1 Addition of Matrices

Two matrices are conformable for addition if they are of the

The sum $A + B$ of two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the $m \times n$ matrix $C = [c_{ij}]$ formed by adding the corresponding entries of A and B together. In symbols, we write as $C = A + B$, that is: $[c_{ij}] = [a_{ij} + b_{ij}]$
where $c_{ij} = a_{ij} + b_{ij}$ for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$.

Note that $a_{ij} + b_{ij}$ is the (i, j) th element of $A + B$.

Transpose of a Matrix: If A is a matrix of order $m \times n$ then an $n \times m$ matrix obtained by interchanging the rows and columns of A, is called the transpose of A. It is denoted by A'. If $A = [a_{ij}]_{m\times n}$, then the transpose of A is defined as:

$$A^t = [a'_{ij}]_{n\times m} \text{ where } a'_{ij} = a_{ji} \text{ for } i = 1, 2, 3, \ldots, n \text{ and } j = 1, 2, 3, \ldots, m$$

For example, if $B = [b_{ij}]_{3\times 4} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$, then

$B^t = [b'_{ij}]_{4\times 3}$ where $b'_{ij} = b_{ji}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ i.e.,

$$B^t = \begin{bmatrix} b'_{11} & b'_{12} & b'_{13} \\ b'_{21} & b'_{22} & b'_{23} \\ b'_{31} & b'_{32} & b'_{33} \\ b'_{41} & b'_{42} & b'_{43} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \\ b_{14} & b_{24} & b_{34} \end{bmatrix}$$

Note that the 2nd row of B has the same entries respectively as the 2nd column of B^t and the 3rd row of B^t has the same entries respectively as the 3rd column of B etc.

Example 1:

If $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix}$, then show that

$$(A + B)^t = A^t + B^t$$

$M_{m \times n}$ possesses the closure property with respect to scalar multiplication. If $A, B \in M_{m \times n}$ and r, s are scalars, then we can prove that $r(sA) = (rs)A$, $(r+s)A = rA + sA$, $r(A+B) = rA + rB$.

3.1.3 Subtraction of Matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order $m \times n$, then we define subtraction of B from A as:

$$
\begin{aligned}
A - B &= A + (-B) \\
&= [a_{ij}] + [-b_{ij}] \\
&= [a_{ij} + (-b_{ij})] = [a_{ij} - b_{ij}] \qquad \text{for } i = 1, 2, 3, \ldots, m;\ j = 1, 2, 3, \ldots, n
\end{aligned}
$$

Thus the matrix $A-B$ is formed by subtracting each entry of B from the corresponding entry of A.

3.1.4 Multiplication of two Matrices

Two matrices A and B are said to be conformable for the product AB if the number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]$ be a 2×3 matrix and $B = [b_{ij}]$ be a 3×2 matrix. Then the product AB is defined to be the 2×2 matrix C whose element c_{ij} is the sum of products of the corresponding elements of the ith row of A with elements of jth column of B. The element c_{21} of C is shown in the figure (A), that is

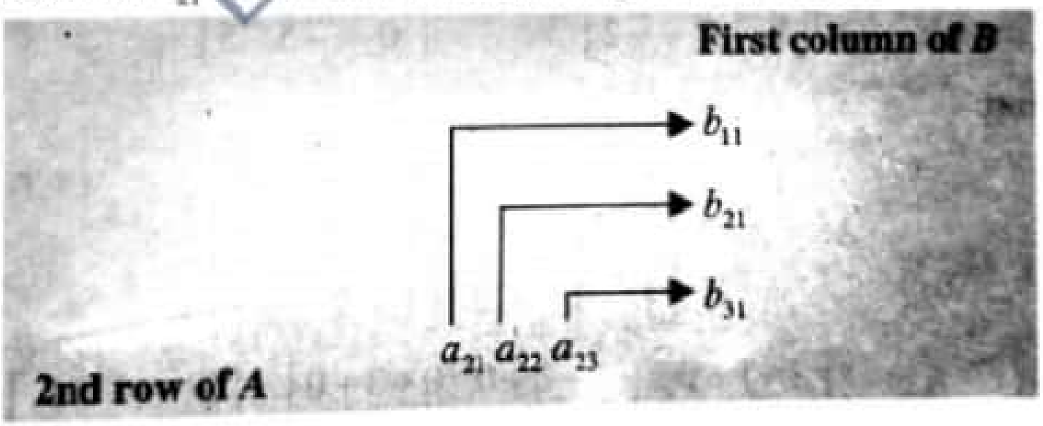

Fig.(A)

$c_{21} = a_{21} b_{11} + a_{22} b_{21} + a_{23} b_{31}$. Thus

$$
AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}
$$

$$= \begin{bmatrix} 6+4+0 & -6+16-15 & 9-24+15 \\ 2+3+0 & -2+12+0 & 3-18+0 \\ 4+1+0 & -4+4+10 & 6-6-10 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 0 \\ 5 & 10 & -15 \\ 5 & 10 & -10 \end{bmatrix}$$

Note: Powers of square matrices are defined as:

$$A^2 = A \times A,\ A^3 = A \times A \times A,$$

$$A^n = A \times A \times A \times \ldots \text{ to } n \text{ factors.}$$

3.2 Determinant of a 2×2 matrix

We can associate with every square matrix A over $\mathcal{R}$ or C, a number $|A|$, known as the determinant of the matrix A.

The determinant of a matrix is denoted by enclosing its square array between vertical bars instead of brackets. The number of elements in any row or column is called the order of determinant. For example, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant of A is $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Its value is defined to be the real number $ad - bc$, that is,

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example, if $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$, then

$$|A| = \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = (2)\,(3) - (-1)(4) = 6 + 4 = 10$$

and $$|B| = \begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix} = (1)\,(8) - (4)(2) = 8 - 8 = 0$$

Hence the determinant of a matrix is the difference of the products of the entries in the two diagonals.

$-bc \quad ad$

Note: The determinant of a 1×1 matrix $[a_{11}]$ is defined as $|a_{11}| = a_{11}$

3.2.1 Singular and Non-Singular Matrices

A square matrix A is ***singular*** if $|A| = 0$, otherwise it is a ***non-singular*** matrix

In the above example, $|B| = 0 \Rightarrow B = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$ is a singular matrix

and $|A| = 10 \neq 0 \Rightarrow A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ is a non-singular matrix.

3.2.2 Adjoint of a 2 × 2 Matrix

The adjoint of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by *adj A* and is defined as:

$$adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

3.2.3 Inverse of a 2 × 2 Matrix

Let A be a non-singualr square matrix of order 2. If there exists a matrix B such that $AB = BA = I_2$ where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then B is called the multiplicative inverse of A and is usually denoted by A^{-1}, that is,

$$B = A^{-1}$$

$$\text{Thus} \quad AA^{-1} = A^{-1}A = I_2$$

Example 3: For a non-singular matrix A, prove that $A^{-1} = \frac{1}{|A|} adj\ A$

Solution: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, Then:

$AA^{-1} = I_2$, that is,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or $\begin{bmatrix} ap + br & aq + bs \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$\Rightarrow \quad x_1 = 1$ and $x_2 = 2$

(ii) The matrix form of the system $\left.\begin{array}{r} x_1 + 2x_2 = 4 \\ 2x_1 + 4x_2 = 12 \end{array}\right\}$ is

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$$

and $\quad |A| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$, so A^{-1} does not exist.

Multiplying the first equation of the above system by 2, we have

$$2x_1 + 4x_2 = 8 \text{ but } 2x_1 + 4x_2 = 12$$

which is impossible. Thus the system has no solution.

Exercise 3.1

1. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix}$, then show that

   i) $4A - 3A = A$ ii) $3B - 3A = 3(B - A)$

2. If $\quad A = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}$, show that $A^4 = I_2$.

3. Find x and y if

   i) $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} y & 1 \\ -3 & 2x \end{bmatrix}$

4. If $\quad A = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix}$, find the following matrices;

   i) $4A - 3B$ ii) $A + 3(B - A)$

5. Find x and y if $\begin{bmatrix} 2 & 0 & x \\ 1 & y & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & x & y \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$

6. If $A = [a_{ij}]_{3\times3}$, show that

   i) $\lambda(\mu A) = (\lambda\mu)A$ ii) $(\lambda + \mu)A = \lambda A + \mu A$ iii) $\lambda A - A = (\lambda -$

$$=\begin{bmatrix} 4+1+9+0 & 2+0+12+0 & -6-5+6+0 \\ 2+0+12+0 & 1+0+16+4 & -3+0+8+2 \\ -6-5+6+0 & -3+0+8+2 & 9+25+4+1 \end{bmatrix}$$

$$=\begin{bmatrix} 14 & 14 & -5 \\ 14 & 21 & 7 \\ -5 & 7 & 39 \end{bmatrix}$$

As $\quad A'=\begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$, so $(A')' = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$ which is A,

That is, $(A')' = A$. (Note that this rule holds for any matrix A.)

Exercise 3.2

1. If $A=[a_{ij}]_{3\times 4}$, then show that

i) $I_3A = A$ ii) $AI_4 = A$

2. Find the inverses of the following matrices.

i) $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$ iv) $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

3. Solve the following system of linear equations.

i) $\begin{matrix} 2x_1 - 3x_2 = 5 \\ 5x_1 + x_2 = 4 \end{matrix}$ ii) $\left.\begin{matrix} 4x_1 + 3x_2 = 5 \\ 3x_1 - x_2 = 7 \end{matrix}\right\}$ iii) $\left.\begin{matrix} 3x - 5y = 1 \\ -2x + y = -3 \end{matrix}\right\}$

4. If $A=\begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$ and $C=\begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$, then find

i) $A-B$ ii) $B-A$ iii) $(A-B)-C$ iv) $A-(B-C)$

5. If $A=\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}$, $B=\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}$ and $C=\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$, then show that

i) $(AB)C = A(BC)$ ii) $(A+B)C = AC + BC$

6. If A and B are square matrices of the same order, then explain why in general

i) $(A+B)^2 \neq A^2 + 2AB + B^2$ ii) $(A-B)^2 \neq A^2 - 2AB + B^2$

iii) $(A+B)(A-B) \neq A^2 - B^2$

Exercise 3.3

Evaluate the following determinants.

1. i) $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$ ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$ iii) $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$

iv) $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$ v) $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$ vi) $\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$

2. Without expansion show that

i) $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$ ii) $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$ iii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

3. Show that

i) $\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$

ii) $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix} = 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$ iii) $\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l^2(3a+l)$

iv) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ v) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

vi) $\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = a^3+b^3$ vii) $\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix} = r$

viii) $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3+b^3+c^3-3abc$

ix) $\begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix} = \lambda^2(a+b+c+\lambda)$

Exercise 3.5

1. Solve the following systems of linear equations by Cramer's rule.

i) $\left.\begin{array}{r} 2x+2y+z=3 \\ 3x-2y-2z=1 \\ 5x+y-3z=2 \end{array}\right\}$ ii) $\left.\begin{array}{r} 2x_1-x_2+x_3=5 \\ 4x_1+2x_2+3x_3=8 \\ 3x_1-4x_2-x_3=3 \end{array}\right\}$ iii) $\left.\begin{array}{r} 2x_1-x_2+x_3=8 \\ x_1+2x_2+2x_3=6 \\ x_1-2x_2-x_3=1 \end{array}\right\}$

2. Use matrices to solve the following systems:

i) $\left.\begin{array}{r} x-2y+z=-1 \\ 3x+y-2z=4 \\ y-z=1 \end{array}\right\}$ ii) $\left.\begin{array}{r} 2x_1+x_2+3x_3=3 \\ x_1+x_2-2x_3=0 \\ -3x_1-x_2+2x_3=-4 \end{array}\right\}$ iii) $\left.\begin{array}{r} x+y=2 \\ 2x-z=1 \\ 2y-3z=-1 \end{array}\right\}$

3. Solve the following systems by reducing their augmented matrices to the echelon form and the reduced echelon forms.

i) $\left.\begin{array}{r} x_1-2x_2-2x_3=-1 \\ 2x_1+3x_2+x_3=1 \\ 5x_1-4x_2-3x_3=1 \end{array}\right\}$ ii) $\left.\begin{array}{r} x+2y+z=2 \\ 2x+y+2z=-1 \\ 2x+3y-z=9 \end{array}\right\}$ iii) $\left.\begin{array}{r} x_1+4x_2+2x_3=2 \\ 2x_1+x_2-2x_3=9 \\ 3x_1+2x_2-2x_3=12 \end{array}\right\}$

4. Solve the following systems of homogeneous linear equations.

i) $\left.\begin{array}{r} x+2y-2z=0 \\ 2x+y+5z=0 \\ 5x+4y+8z=0 \end{array}\right\}$ ii) $\left.\begin{array}{r} x_1+4x_2+2x_3=0 \\ 2x_1+x_2-3x_3=0 \\ 3x_1+2x_2-4x_3=0 \end{array}\right\}$ iii) $\left.\begin{array}{r} x_1-2x_2-x_3=0 \\ x_1+x_2+5x_3=0 \\ 2x_1-x_2+4x_3=0 \end{array}\right\}$

5. Find the value of λ for which the following systems have non-trivial solution. Also solve the system for the value of λ.

i) $\left.\begin{array}{r} x+y+z=0 \\ 2x+y-\lambda z=0 \\ x+2y-2z=0 \end{array}\right\}$ ii) $\left.\begin{array}{r} x_1+4x_2+\lambda x_3=0 \\ 2x_1+x_2-3x_3=0 \\ 3x_1+\lambda x_2-4x_3=0 \end{array}\right\}$

6. Find the value of λ for which the following system does not possess a unique solution. Also solve the system for the value of λ.

$$\left.\begin{array}{r} x_1+4x_2+\lambda x_3=2 \\ 2x_1+x_2-2x_3=11 \\ 3x_1+2x_2-2x_3=16 \end{array}\right\}$$