1 Number Systems

1.1 Introduction

In the very beginning, human life was simple. An early ancient herdsman compared sheep (or cattle) of his herd with a pile of stones when the herd left for grazing and again on its return for missing animals. In the earliest systems probably the vertical strokes or bars such as I, II, III, IIII etc., were used for the numbers 1, 2, 3, 4 etc. The symbol "IIIII" was used by many people including the ancient Egyptians for the number of fingers of one hand.

Around 5000 B.C, the Egyptians had a number system based on 10. The symbol ∩ for 10 and 𓍢 for 100 were used by them. A symbol was repeated as many times as it was needed. For example, the numbers 13 and 324 were symbolized as ∩III and 𓍢𓍢𓍢∩∩IIII respectively. The symbol 𓍢𓍢𓍢∩∩IIII was interpreted as 100+100+100+10+10+1+1+1+1. Different people invented their own symbols for numbers. But these systems of notations proved to be inadequate with advancement of societies and were discarded. Ultimately the set {1, 2, 3, 4, ...} with base 10 was adopted as the counting set (also called the set of natural numbers). The solution of the equation $x+2=2$ was not possible in the set of natural numbers, so the natural number system was extended to the set of whole numbers. No number in the set of whole numbers W could satisfy the equation $x+4=2$ or $x+a=b$, if $a>b$, and $a,b\in W$. The negative integers −1, −2, −3, ... were introduced to form the set of integers $Z=\{0,\pm 1,\pm 2,...\}$.

Again the equation of the type $2x=3$ or $bx=a$ where $a,b\in Z$ and $b\neq 0$ had no solution in the set Z, so the numbers of the form $\frac{a}{b}$ where $a,b\in Z$ and $b\neq 0$, were invented to remove such difficulties. The set $Q=\{\frac{a}{b} \mid a,b\in Z \wedge b\neq 0\}$ was named as the set of rational numbers. Still the solution of equations such as $x^2=2$ or $x^2=a$ (where a is not a perfect square) was not possible in the set Q. So the irrational numbers of the type $\pm\sqrt{2}$ or $\pm\sqrt{a}$ where a is not a perfect square we introduced. This process of enlargement of the number system ultimately led to t set of real numbers $\mathcal{R} = Q \cup Q'$ (Q' is the set of irrational numbers) which is us most frequently in everyday life.

1.2 Rational Numbers and Irrational Numbers

We know that a **rational number** is a number which can be put in the form

where $p, q \in Z \wedge q \neq 0$. The numbers $\sqrt{16}$, 3.7,4 etc., are rational numbers. $\sqrt{16}$ c be reduced to the form $\frac{p}{q}$ where $p, q \in Z$, and $q \neq 0$ because $\sqrt{16} = 4 = \frac{4}{1}$.

Irrational numbers are those numbers which cannot be put into the form

where $p, q \in Z$ and $q \neq 0$. The numbers $\sqrt{2}, \sqrt{3}, \frac{7}{\sqrt{5}}, \sqrt{\frac{5}{16}}$ are irrational numbers.

1.2.1 Decimal Representation of Rational and Irrational Numbers

1) Terminating decimals: A decimal which has only a finite number of digits in decimal part, is called a terminating decimal. Thus 202.04, 0.0000415, 100000.412378 are examples of terminating decimals.

Since a terminating decimal can be converted into a common fraction, so eve terminating decimal represents a **rational number**.

2) Recurring Decimals: This is another type of **rational numbers**. In general, recurring or periodic decimal is a decimal in which one or more digits repe indefinitely.

It will be shown (in the chapter on sequences and series) that a recurring decim can be converted into a common fraction. So **every recurring decimal represents rational number:**

A non-terminating, non-recurring decimal is a decimal which neither termina nor it is recurring. It is not possible to convert such a decimal into a comm fraction. Thus a **non-terminating, non-recurring decimal represents an irratio number.**

Example 1:

i) $.25 \; (= \frac{25}{100})$ is a rational number.

ii) $.333.... \; (= \frac{1}{3})$ is a recurring decimal, it is a rational number.

iii) $2.\bar{3} (= 2.333...)$ is a rational number.

iv) $0.142857142857 \ldots \; (= \frac{1}{7})$ is a rational number.

v) $0.0100100010000 1\ldots$ is a non-terminating, non-periodic decimal, so it is an irrational number.

vi) 214.121122111222 1111 2222 ... is also an irrational number.

vii) 1.4142135 ... is an irrational number.

viii) 7.3205080 ... is an irrational number.

ix) 1.709975947 ... is an irrational number.

x) 3.141592654... is an important irrational number called $\pi(Pi)$ which denotes the constant ratio of the circumference of any circle to the length of its diameter i.e.,

$$\pi = \frac{\text{circumference of any circle}}{\text{length of its diameter.}}$$

An approximate value of π is $\frac{22}{7}$, a better approximation is $\frac{355}{113}$ and a still better approximation is 3.14159. The value of π correct to 5 lac decimal places has been determined with the help of computer.

Example 2: Prove $\sqrt{2}$ is an irrational number.

Solution: Suppose, if possible, $\sqrt{2}$ is rational so that it can be written in the form p/q where $p, q \in Z$ and $q \neq 0$. Suppose further that p/q is in its lowest form.

Then $\sqrt{2} = p/q$, $\quad (q \neq 0)$

Squaring both sides we get;

$$2 = \frac{p^2}{q^2} \text{ or } p^2 = 2q^2 \qquad (1)$$

The R.H.S. of this equation has a factor 2. Its L.H.S. must have the same fa

Now a prime number can be a factor of a square only if it occurs at least tw the square. Therefore, p^2 should be of the form $4p'^2$ so that equation (1) tak form:

$$4p'^2 = 2q^2 \quad \text{....(2)}$$

i.e., $$2p'^2 = q^2 \quad \text{....(3)}$$

In the last equation, 2 is a factor of the L.H.S. Therefore, q^2 should be form $4q'^2$ so that equation 3 takes the form

$$2p'^2 = 4q'^2 \quad \text{i.e.,} \quad p'^2 = 2q'^2 \quad \text{....(4)}$$

From equations (1) and (2),

$$p = 2p'$$

and from equations (3) and (4),

$$q = 2q'$$

$$\therefore \quad \frac{p}{q} = \frac{2p'}{2q'}$$

This contradicts the hypothesis that $\frac{p}{q}$ is in its lowest form.

Hence $\sqrt{2}$ is irrational.

Example 3: Prove $\sqrt{3}$ is an irrational number.

Solution: Suppose, if possible $\sqrt{3}$ is rational so that it can be written in the p/q when $p, q \in Z$ and $q \neq 0$. Suppose further that p/q is in its lowest form,

then $\sqrt{3} = p/q$, $\quad (q \neq 0)$

Squaring this equation we get;

$$3 = \frac{p^2}{q^2} \text{ or } p^2 = 3q^2 \quad \text{....(1)}$$

of this equation has a factor 3. Its L.H.S. must have the same factor.

Now a prime number can be a factor of a square only if it occurs at least twice in he square. Therefore, p^2 should be of the form $9p'^2$ so that equation (1) takes the orm:

$$9p'^2 = 3q^2 \tag{2}$$

i.e., $3p'^2 = q^2$ (3)

In the last equation, 3 is a factor of the L.H.S. Therefore, q^2 should be of the orm $9q'^2$ so that equation (3) takes the form

$$3p'^2 = 9q'^2 \quad \text{i.e.,} \quad p'^2 = 3q'^2 \tag{4}$$

rom equations (1) and (2),

$$p = 3p'$$

nd from equations (3) and (4)

$$q = 3q'$$

$$\therefore \quad \frac{p}{q} = \frac{3p'}{3q'}$$

This contradicts the hypothesis that $\frac{p}{q}$ is in its lowest form.

Hence $\sqrt{3}$ is irrational.

Note: Using the same method we can prove the irrationality of $\sqrt{5}, \sqrt{7}, \ldots, \sqrt{n}$ where n is any prime number.

1.3 Properties of Real Numbers

We are already familiar with the set of real numbers and most of th properties. We now state them in a unified and systematic manner. Before stati them we give a preliminary definition.

Binary Operation: A binary operation may be defined as a function from $A \times A$ ir A, but for the present discussion, the following definition would serve the purpose.

A *binary operation* in a set A is a rule usually denoted by $*$ that assigns to any pa of elements of A, taken in a definite order, another element of A.

Two important binary operations are addition and multiplication in the set real numbers. Similarly, union and intersection are binary operations on sets whi are subsets of the same Universal set.

$\mathbb{R}$ usually denotes the set of real numbers. We assume that two bina operations addition (+) and multiplication (. or $\times$) are defined in $\mathbb{R}$. Following a the properties or laws for real numbers.

1. Addition Laws: -

i) **Closure Law of Addition**

$\forall\, a, b \in \mathbb{R}, a + b \in \mathbb{R}$ ($\forall$ stands for "for all")

ii) **Associative Law of Addition**

$\forall\, a,b,c \in \mathbb{R}, a + (b + c) = (a+b) + c$

iii) **Additive Identity**

$\forall\, a \in \mathbb{R}, \exists\, 0 \in \mathbb{R}$ such that $a + 0 = 0 + a = a$

($\exists$ stands for "there exists").

0 (read as zero) is called the identity element of addition.

iv) **Additive Inverse**

$\forall\, a \in \mathbb{R}, \exists\, (-a) \in \mathbb{R}$ such that

$a + (-a) = 0 = (-a) + a$

v) **Commutative Law for Addition**

$\forall\, a, b \in \mathbb{R}, a + b = b + a$

2. Multiplication Laws

vi) **Closure Law of Multiplication**

$\forall\, a,b \in \mathbb{R}, a.b \in \mathbb{R}$ ($a.b$ is usually written as ab).

vii) **Associative Law for Multiplication**

$\forall\, a,b,c \in \mathbb{R},\ a\,(bc) = (ab)\,c$

viii) **Multiplicative Identity**

$\forall a \in \mathbb{R},\ \exists\ 1 \in \mathbb{R}$ such that $a.1 = 1.a = a$

1 is called the multiplicative identity of real numbers.

ix) **Multiplicative Inverse**

$\forall a(\neq 0) \in \mathbb{R}, \exists\, a^{-1} \in \mathbb{R}$ such that $a.a^{-1} = a^{-1}.a = 1$ (a^{-1} is also written as $\dfrac{1}{a}$).

x) **Commutative Law of multiplication**

$\forall\, a, b \in \mathbb{R},\ ab = ba$

3. Multiplication – Addition Law

xi) $\forall\, a, b, c, \in \mathbb{R}$

$a\,(b+c) = ab + ac$ (<u>Distributivity</u> of multiplication over addition).

$(a+b)\,c = ac + bc$

In addition to the above properties $\mathbb{R}$ possesses the following properties.

i) Order Properties (described below).

ii) Completeness axiom which will be explained in higher classes.

The above properties characterizes $\mathbb{R}$ i.e., only $\mathbb{R}$ possesses all these properties. Before stating the order axioms we state the properties of equality of numbers.

4. Properties of Equality

Equality of numbers denoted by "=" possesses the following properties: -

i) <u>Reflexive property</u> $\forall\, a \in \mathbb{R},\ a = a$

ii) <u>Symmetric Property</u> $\forall\, a, b \in \mathbb{R},\ a = b \Rightarrow b = a.$

iii) <u>Transitive Property</u> $\forall\, a, b, c \in \mathbb{R},\ a = b \wedge b = c \Rightarrow a = c$

iv) <u>Additive Property</u> $\forall\, a, b, c \in \mathbb{R},\ a = b \Rightarrow a + c = b + c$

v) <u>Multiplicative Property</u> $\forall\, a, b, c \in \mathbb{R},\ a = b \Rightarrow ac = bc \wedge ca = cb.$

vi) <u>Cancellation Property w.r.t. addition</u>

$\forall\, a, b, c \in \mathbb{R},\ a + c = b + c \Rightarrow a = b$

vii) <u>Cancellation Property w.r.t. Multiplication:</u>

$\forall\, a, b, c \in \mathbb{R},\ ac = bc \Rightarrow a = b,\ c \neq 0.$

5. Properties of Inequalities (Order properties)

1) Trichotomy Property $\forall\ a, b \in \mathbb{R}$

either $a = b$ or $a > b$ or $a < b$

2) Transitive Property $\forall\ a, b, c \in \mathbb{R}$

i) $a > b \wedge b > c \Rightarrow a > c$ ii) $a < b \wedge b < c \Rightarrow a < c$

3) Additive Property: $\forall\ a, b, c \in \mathbb{R}$

a) i) $a > b \Rightarrow a + c > b + c$ b) i) $a > b \wedge c > d \Rightarrow a+c > b+d$

ii) $a < b \Rightarrow a + c < b + c$ ii) $a < b \wedge c < d \Rightarrow a+c < b+d$

4) Multiplicative Properties:

a) $\forall\ a, b, c \in \mathbb{R}$ and $c > 0$

i) $a > b \Rightarrow ac > bc$ ii) $a < b \Rightarrow ac < bc$.

b) $\forall\ a, b, c \in \mathbb{R}$ and $c < 0$.

i) $a > b \Rightarrow ac < bc$ ii) $a < b \Rightarrow ac > bc$

c) $\forall\ a, b, c, d \in \mathbb{R}$ and a, b, c, d are all positive.

i) $a > b \wedge c > d \Rightarrow ac > bd$. ii) $a < b \wedge c < d \Rightarrow ac < bd$.

Note That:

Note: **1.** Any set possessing all the above 11 properties is called a field.

2. From the multiplicative properties of inequality we conclude that: -
If both the sides of an inequality are multiplied by a +ve number, it direction does not change, but multiplication of the two sides by –ve number reverses the direction of the inequality.

3. a and $(-a)$ are additive inverses of each other.
Since by definition inverse of $-a$ is a,

$$\therefore \quad -(-a) = a$$

4. The left hand member of the above equation should be read as negative of 'negative a' and not ' minus minus a'.

5. a and $\frac{1}{a}$ are the multiplicative inverses of each other. Since by definition inverse of $\frac{1}{a}$ is a (i.e., inverse of a^{-1} is a)

$$\therefore \quad (a^{-1})^{-1} = a \quad \text{or} \quad \frac{1}{\frac{1}{a}} = a$$

Example 4: Prove that for any real numbers a, b

i) $a.0 = 0$ ii) $ab = 0 \Rightarrow a = 0 \vee b = 0$ [$\vee$ stands for "or"]

Solution: i) $a.0 = a\,[1 + (-1)]$ (Property of additive inverse)

$= a\,(1 - 1)$ (Def. of subtraction)

$= a.1 - a.1$ (Distributive Law)

$= a - a$ (Property of multiplicative identity)

$= a + (-a)$ (Def. of subtraction)

$= 0$ (Property of additive inverse)

Thus $a.0 = 0$.

(ii) Given that $ab = 0$ (1)

Suppose $a \neq 0$, then $\frac{1}{a}$ exists

(1) gives: $\frac{1}{a}\,(ab) = \frac{1}{a}.0$ (Multiplicative property of equality)

$\Rightarrow (\frac{1}{a}.a)\,b = \frac{1}{a}.0$ (Assoc. Law of $\times$)

$\Rightarrow 1.b = 0$ (Property of multiplicative inverse).

$\Rightarrow b = 0$ (Property of multiplicative identity).

Thus if $ab = 0$ and $a \neq 0$, then $b = 0$

Similarly it may be shown that

if $ab = 0$ and $b \neq 0$, then $a = 0$.

Hence $ab = 0 \Rightarrow a = 0$ or $b = 0$.

Example 5: For real numbers a,b show the following by stating the properties used.

i) $(-a)\,b = a\,(-b) = -ab$ ii) $(-a)\,(-b) = ab$

Solution: i) $(-a)(b) + ab = (-a + a)\,b$ (Distributive law)

$= 0.b = 0.$ (Property of additive inverse)

$\therefore (-a)b + ab = 0$

i.e., $(-a)b$ and ab are additive inverse of each other.

$\therefore (-a)b = -(ab) = -ab$ ($\because -(ab)$ is written as $-ab$)

ii) $(-a)\,(-b) - ab = (-a)(-b) + (-ab)$

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$\Rightarrow b = 0$ (Property of multiplicative identity).

Thus if $ab = 0$ and $a \neq 0$, then $b = 0$

Similarly it may be shown that

if $ab = 0$ and $b \neq 0$, then $a = 0$.

Hence $ab = 0 \Rightarrow a = 0$ or $b = 0$.

Example 5: For real numbers a, b show the following by stating the properties used.

i) $(-a)\,b = a\,(-b) = -ab$ ii) $(-a)(-b) = ab$

Solution: i) $(-a)(b) + ab = (-a + a)\,b$ (Distributive law)

$= 0.b = 0.$ (Property of additive inverse)

$\therefore (-a)b + ab = 0$

i.e., $(-a)b$ and ab are additive inverse of each other.

$\therefore (-a)b = -(ab) = -ab$ ($\because -(ab)$ is written as $-ab$)

ii) $(-a)(-b) - ab = (-a)(-b) + (-ab)$

$= (-a)(-b) + (-a)(b)$ (By (i))

$= (-a)(-b + b)$ (Distributive law)

$= (-a).0 = 0.$ (Property of additive inverse)

$(-a)(-b) = ab$

Example 6: Prove that

i) $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$ (Principle for equality of fractions)

ii) $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$

iii) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ (Rule for product of fractions).

iv) $\frac{a}{b} = \frac{ka}{kb}, (k \neq 0)$ (Golden rule of fractions)

v) $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$ (Rule for quotient of fractions).

The symbol $\Leftrightarrow$ stands for iff i.e., if and only if.

Solution:

i) $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b}(bd) = \frac{c}{d}(bd)$

$\Rightarrow \frac{a.1}{b}(bd) = \frac{c.1}{d}(bd)$

$\Rightarrow a.(\frac{1}{b}.b).\, d = c.(\frac{1}{d}\, .\, bd)$

$= c\ (bd\, .\, \frac{1}{d})$

$\Rightarrow ad = cb$

$\therefore ad = bc$

Again $ad = bc \Rightarrow (ad) \times \frac{1}{b} . \frac{1}{d} = b.c.\ \frac{1}{b} . \frac{1}{d}$

$\Rightarrow a.\frac{1}{b}.d\frac{1}{d} \quad = b.\frac{1}{b}\ .\ c.\frac{1}{d}$

$$\Rightarrow \frac{a}{b} = \frac{c}{d}.$$

ii) $(ab).\frac{1}{a}.\frac{1}{b} = (a.\frac{1}{a}).(b.\frac{1}{b}) = 1.1 = 1$

Thus ab and $\frac{1}{a}.\frac{1}{b}$ are the multiplicative inverse of each other. But multiplicative inverse of ab is $\frac{1}{ab}$

$$\therefore \frac{1}{ab} = \frac{1}{a}.\frac{1}{b}.$$

iii) $\frac{a}{b}.\frac{c}{d} = (a.\frac{1}{b}).(c.\frac{1}{d})$

$= (ac)\ (\frac{1}{b}.\frac{1}{d})$ (Using commutative and associative laws of multiplication)

$= ac.\frac{1}{bd} = \frac{ac}{bd}.$

$= \frac{a}{b}.\frac{c}{d} = \left|\frac{ac}{bd}\right|$

iv) $\frac{a}{b} = \frac{a}{b}.1 = \frac{a}{b}.\frac{k}{k} = \frac{ak}{ak}$

$\therefore \frac{a}{b} = \frac{ak}{bk}.$

v) $$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}(bd)}{\frac{c}{d}(bd)} = \frac{ad(\frac{1}{b}b)}{cb(\frac{1}{d}d)} = \frac{ad}{bc}.$$

Example 7 : Does the set $\{1, -1\}$ possess closure property with respect to

i) addition ii) multiplication?

Solution: i) $1+1 = 2,\ 1+(-1) = 0 = -1+1$

$-1+(-1) = -2$

But 2, 0, −2 do not belong to the given set. That is, all the sums do not belong to the given set. So it does not possess closure property w.r.t. addition.

ii) $1.1=1$, $1.(-1)=-1$, $(-1).1=-1$, $(-1).(-1)=1$

Since all the products belong to the given set, it is closed w.r.t. multiplication.

Exercise 1.1

1. Which of the following sets have closure property w.r.t. addition and multiplication?

   i) $\{0\}$ ii) $\{1\}$ iii) $\{0,-1\}$ iv) $\{1,-1\}$

2. Name the properties used in the following equations. (Letters, where used, represent real numbers).

   i) $4+9=9+4$
   ii) $(a+1)+\frac{3}{4}=a+(1+\frac{3}{4})$
   iii) $(\sqrt{3}+\sqrt{5})+\sqrt{7}=\sqrt{3}+(\sqrt{5}+\sqrt{7})$
   iv) $100+0=100$
   v) $1000\times 1 = 1000$
   vi) $4.1+(-4.1)=0$
   vii) $a-a=0$
   viii) $\sqrt{2}\times\sqrt{5} = \sqrt{5}\times\sqrt{2}$
   ix) $a(b-c)=ab-ac$
   x) $(x-y)z=xz-yz$
   xi) $4\times(5\times 8)=(4\times 5)\times 8$
   xii) $a(b+c-d)=ab+ac-ad$.

3. Name the properties used in the following inequalities:

   i) $-3<-2 \Rightarrow 0<1$
   ii) $-5<-4 \Rightarrow 20>16$
   iii) $1>-1 \Rightarrow -3>-5$
   iv) $a<0 \Rightarrow -a>0$
   v) $a>b \Rightarrow \frac{1}{a}<\frac{1}{b}$
   vi) $a>b \Rightarrow -a<-b$

4. Prove the following rules of addition: -

   i) $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$
   ii) $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$

5. Prove that $-\frac{7}{12}-\frac{5}{18}=\frac{-21-10}{36}$

6. Simplify by justifying each step: -

   i) $\frac{4+16x}{4}$
   ii) $\frac{\frac{1}{4}+\frac{1}{5}}{\frac{1}{4}-\frac{1}{5}}$
   iii) $\frac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}$
   iv) $\frac{\frac{1}{a}-\frac{1}{b}}{1-\frac{1}{a}.\frac{1}{b}}$

1.4 Complex Numbers

The history of mathematics shows that man has been developing and enlarging his concept of **number** according to the saying that "Necessity is the mother of invention". In the remote past they stared with the set of counting numbers and invented, by stages, the negative numbers, rational numbers irrational numbers. Since square of a positive as well as negative number is a positive number, the square root of a negative number does not exist in the realm of real numbers. Therefore, square roots of negative numbers were given no attention for centuries together. However, recently, properties of numbers involving square roots of negative numbers have also been discussed in detail and such numbers have been found useful and have been applied in many branches of pure and applied mathematics. The numbers of the form $x+iy$, where $x, y \in \mathcal{R}$ and $i=\sqrt{-1}$, are called **complex numbers,** here x is called **real part** and y is called **imaginary part** of the complex number. For example, $3+4i$, $2-\frac{5}{7}i$ etc. are complex numbers.

Note: Every real number is a complex number with 0 as its imaginary part.

Let us start with considering the equation.

$$x^2+1=0 \qquad (1)$$

$$\Rightarrow \quad x^2=-1$$

$$\Rightarrow \quad x=\pm\sqrt{-1}$$

$\sqrt{-1}$ does not belong to the set of real numbers. We, therefore, for convenience call it **imaginary number** and denote it by i (read as iota).

The product of a real number and i is also an **imaginary number**. Thus $2i$, $-3i$, $\sqrt{5}\,i$, $-\frac{11}{2}i$ are all imaginary numbers. i which may be written $1.i$ is also an imaginary number.

Powers of *i* :

$$i^2=-1 \quad \text{(by definition)}$$

$$i^3=i^2\,i=-1.i=-i$$

$$i^4=i^2\times i^2=(-1)(-1)=1$$

Thus any power of i must be equal to $1, i, -1$ or $-i$. For instance,

$$i^{13} = (i^2)^6\, i = (-1)^6\, i = i$$
$$i^6 = (i^2)^3 = (-1)^3 = -1 \quad \text{etc.}$$

1.4.1 Operations on Complex Numbers

With a view to develop algebra of **complex numbers**, we state a few definitions. The symbols a, b, c, d, k, where used, represent real numbers.

1) $a + bi = c + di \Rightarrow a = c \wedge b = d$.
2) Addition: $(a+bi)+(c+di)=(a+c)+(b+d)i$
3) $k(a+bi) = ka + kbi$
4) $(a+bi) - (c+di) = (a+bi) + [-(c+di)]$
$$= a + bi + (-c - d\,i)$$
$$= (a-c) + (b-d)i$$
5) $(a+bi).(c+di) = ac + adi + bci + bd\,i^2 = (ac - bd) + (ad + bc)i$.
6) **Conjugate Complex Numbers:** *Complex numbers* of the form $(a+bi)$ and $(a-bi)$ which have the same real parts and whose imaginary parts differ in sign only, are called conjugates of each other. Thus $5 + 4i$ and $5 - 4i$, $-2 + 3i$ and $-2-3i$, $-\sqrt{5}\, i$ and $\sqrt{5}\, i$ are three pairs of conjugate numbers.

Note: A real number is self-conjugate.

1.4.2 Complex Numbers as Ordered Pairs of Real Numbers

We can define complex numbers also by using ordered pairs.

Let C be the set of ordered pairs belonging to $\mathbb{R} \times \mathbb{R}$ which are subject to the following properties: -

i) $(a,b) = (c,d) \Leftrightarrow a = c \wedge b = d$.
ii) $(a,b) + (c,d) = (a+c, b+d)$
iii) If k is any real number, then $k(a,b) = (ka, kb)$
iv) $(a,b)\,(c,d) = (ac - bd,\ ad + bc)$

Then C is called the set of *complex numbers*. It is easy to see that

$(a,b) - (c,d) = (a-c,\ b-d)$

Properties (1), (2) and (4) respectively define equality, sum and product of two complex numbers. Property (3) defines the product of a real number and a complex number.

Example 1: Find the sum, difference and product of the complex numbers (8,9) and (5,–6)

Solution: Sum = $(8+5, 9-6) = (13,3)$

Difference $= (8-5, 9-(-6)) = (3,15)$

Product $= (8.5-(9)(-6), 9.5+(-6)8)$

$= (40+54, 45-48)$

$= (94, -3)$

1.4.3 Properties of the Fundamental Operations on Complex Numbers

It can be easily verified that the set C satisfies all the field axioms i.e., it possesses the properties 1(i to v), 2 (vi to x) and 3(xi) of Art. 1.2.

By way of explanation of some points we observe as follows: -

i) The additive identity in C is $(0, 0)$.

ii) Every complex number (a, b) has the additive inverse $(-a, -b)$ i.e., $(a, b)+(-a, -b) = (0,0)$.

iii) The multiplicative identity is $(1,0)$ i.e.,
$(a,b).(1,0) = (a.1-b.0,\ b.1+a.0) = (a, b)$.
$=(1,0)(a, b)$

iv) Every non-zero complex number {i.e., number not equal to (0,0)} has a multiplicative inverse.

The multiplicative inverse of (a, b) is $\left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right)$

$$(a, b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1,0), \text{ the identity element}$$

$$= \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)(a,b)$$

v) $(a, b)[(c, d) \pm (e, f)] = (a, b)(c, d) \pm (a, b)(e, f)$

Note: The set C of complex numbers does not satisfy the order axioms. In fact there is no sense in saying that one complex number is greater or less than another.

2 Sets, Functions and Groups

2.1 Introduction

We are familiar with the notion of a **set** since the word is frequently used in everyday speech, for instance, water set, tea set, sofa set. It is a wonder that mathematicians have developed this ordinary word into a mathematical concept as much as it has become a language which is employed in most branches of modern mathematics.

For the purposes of mathematics, a *set* is generally described as a **well-defined collection of distinct objects**. By a *well-defined collection* is meant a collection, which is such that, given any object, we may be able to decide whether the object belongs to the collection or not. By *distinct objects* we mean objects no two of which are identical (same).

The objects in a set are called its **members** or **elements**. Capital letters A, B, C, X, Y, Z etc., are generally used as names of sets and small letters a, b, c, x, y, z etc., are used as *members* of sets.

There are three different ways of describing a set.

i) **The Descriptive Method:** A set may be described in words. For instance, the set of all vowels of the English alphabets.

ii) **The Tabular Method:** A set may be described by listing its elements within brackets. If A is the set mentioned above, then we may write:

$$A = \{a, e, i, o, u\}.$$

iii) **Set-builder method:** It is sometimes more convenient or useful to employ the method of set-builder notation in specifying sets. This is done by using a symbol or letter for an arbitrary member of the set and stating the property common to all the members. Thus the above set may be written as:

$A = \{x \mid x \text{ is a vowel of the English alphabet}\}$

This is read as A is the set of all x such that x is a vowel of the En[glish] alphabet.

The symbol used for **membership** of a set is $\in$. Thus $a \in A$ means **a i[s an] element of A** or **a belongs to A**. $c \notin A$ means **c does not belong to A** or **c is n[ot a] member of A**. Elements of a set can be anything: people, countries, rivers, objec[ts of] our thought. In algebra we usually deal with sets of numbers. Such sets, along [with] their names are given below: -

N = The set of all natural numbers $= \{1, 2, 3, ...\}$

W = The set of all whole numbers $= \{0, 1, 2, ...\}$

Z = The set of all integers $= \{0, \pm 1, \pm 2, ...\}$.

Z' = The set of all negative integers $= \{-1, -2, -3, ...\}$

O = The set of all odd integers $= \{\pm 1, \pm 3, \pm 5, ...\}$.

E = The set of all even integers $= \{0, \pm 2, \pm 4, ...\}$.

Q = The set of all rational numbers $= \left\{x \mid x = \frac{p}{q} \text{ where } p, q \in Z \text{ and } q \neq 0\right\}$

Q' = The set of all irrational numbers $= \left\{x \mid x \neq \frac{p}{q}, \text{ where } p, q \in Z \text{ and } q \neq 0\right\}$

$\mathcal{R}$ = The set of all real numbers $= Q \cup Q'$

Equal Sets: Two sets A and B are equal i.e., $A = B$, if and only if they have the sa[me] elements that is, if and only if every element of each set is an element of the other s[et].

Thus the sets $\{1, 2, 3\}$ and $\{2, 1, 3\}$ are equal. From the definition of equal[ity] of sets it follows that a mere change in the order of the elements of a set does not al[ter] the set. In other words, while describing a set in the tabular form its elements may [be] written in any order.

Note: (1) $A = B$ **if and only if they have the same elements means if $A = B$ they h[ave] the same elements and if A and B have the same elements then $A = B$.**

(2) The phrase if and only if is shortly written as "iff".

Equivalent Sets: If the elements of two sets A and B can be paired in such a way [that] each element of A is paired with one and only one element of B and vice versa,

such a pairing is called a one-to-one correspondence between A and B e.g., if A = {Billal, Ahsan, Jehanzeb} and B = {Fatima, Ummara, Samina}
then six different (1 – 1) correspondences can be established between A and B.
Two of these correspondences are shown below: -

i). {Billal, Ahsan, Jehanzeb}
$\updownarrow \quad \updownarrow \quad \updownarrow$
{Fatima, Ummara, Samina}

ii). {Billal, Ahsan, Jehanzeb}
$\updownarrow \quad \updownarrow \quad \updownarrow$
{Fatima, Samina, Ummara}

(Write down the remaining 4 correspondences yourselves)

Two sets are said to be equivalent if a (1 – 1) correspondence can be established between them. In the above example A and B are equivalent sets.

Example 1: Consider the sets $N = \{1, 2, 3, \ldots\}$ and $O = \{1, 3, 5, \ldots\}$

We may establish (1–1) correspondence between them in the following manner:

$$\begin{array}{cccccc} \{1, & 2, & 3, & 4, & 5, & \ldots\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \{1, & 3, & 5, & 7, & 9, & \ldots\} \end{array}$$

Thus the sets N and O are equivalent. But notice that they are not equal.

Remember that two equal sets are necessarily equivalent, but the converse may not be true i.e.. two equivalent sets are not necessarily equal.

Sometimes, the symbol ~ is used to mean **is equivalent to**. Thus $N \sim O$.

Order of a Set: There is no restriction on the number of members of a set. A set may have 0, 1, 2, 3 or any number of elements. Sets with zero or one element deserve special attention. According to the everyday use of the word set or collection it must have at least two elements. But in mathematics it is found convenient and useful to consider sets which have only one element or no element at all.

A set having only one element is called a **singleton set** and a set with no elements (zero number of elements) is called the **empty set** or **null set**. The empty set is denoted by the symbol ϕ or { }. The set of odd integers between 2 and 4 is a singleton i.e., the set {3} and the set of even integers between the same numb[ers is] the empty set.

The solution set of the equation $x^2+1=0$, in the set of real numbers is the empty set. Clearly the set $\{0\}$ is a *singleton set* having zero as its only element, not the empty set.

Finite and Infinite sets: If a set is equivalent to the set $\{1, 2, 3, ..., n\}$ for a fixed natural number n, then the set is said to be *finite* otherwise *infinite*.

Sets of number N, Z, Z' etc., mentioned earlier are infinite sets.

The set $\{1, 3, 5, ..., 9999\}$ is a finite set but the set $\{1, 3, 5, ...\}$, which is set of all positive odd natural numbers is an infinite set.

Subset: If every element of a set A is an element of set B, then A is a *subset* of B. Symbolically this is written as: $A \subseteq B$ (A is subset of B)

In such a case we say B is a super set of A. Symbolically this is written as:

$$B \supseteq A \text{ } (B \text{ is a superset of } A)$$

Note: The above definition may also be stated as follows:

$$A \subseteq B \text{ iff } x \in A \Rightarrow x \in B$$

Proper Subset: If A is a subset of B and B contains at least one element which is not an element of A, then A is said to be a *proper subset* of B. In such a case we write: $A \subset B$ (A is a proper subset of B).

Improper Subset: If A is subset of B and $A = B$, then we say that A is an improper subset of B. From this definition it also follows that every set A is an improper subset of itself.

Example 2: Let $A = \{a, b, c\}$, $B = \{c, a, b\}$ and $C = \{a, b, c, d\}$, then clearly $A \subset C$, $B \subset C$ but $A = B$ and $B = A$.

Notice that each of A and B is an improper subset of the other because $A = B$.

Note: When we do not want to distinguish between proper and improper subsets, we may use the symbol $\subseteq$ for the relationship.

It is easy to see that: $N \subset Z \subset Q \subset \mathbb{R}$.

Theorem 1.1: The empty set is a subset of every set.

We can convince ourselves about the fact by rewording the definition of subset as follows: -

A is subset of B if it contains no element which is not an element of B.

Obviously an empty set does not contain such element, which is not contained in another set.

Power Set: A set may contain elements, which are sets themselves. For example if: C = Set of classes of a certain school, then elements of C are sets themselves because each class is a set of students. An important set of sets is the *power set* of a given set.

The *power set* of a set S denoted by $P(S)$ is the set containing all the possible subsets of S.

Example 3: If $A = \{a, b\}$, then $P(A) = \{\Phi, \{a\}, \{b\}, \{a,b\}\}$

Recall that the empty set is a subset of every set and every set is its own subset.

Example 4: If $B = \{1, 2, 3\}$, then

$P(B) = \{\Phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Example 5: If $C = \{a, b, c, d\}$, then

$P(C) = \{\Phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\},$
$\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\}$.

Example 6: If $D = \{a\}$, then $P(D) = \{\Phi, \{a\}\}$

Example 7: If $E = \{\ \}$, then $P(E) = \{\Phi\}$.

Note (1) The power set of the empty set is not empty.

(2) Let $n(S)$ denoted the number of elements of a set S, then $n\{P(S)\}$ denotes the number of elements of the power set of S. From examples 3 to 7 we get the following table of results:

$n(S)$	0	1	2	3	4	5
$n\{P(S)\}$	$1=2^0$	$2=2^1$	$4=2^2$	$8=2^3$	$16=2^4$	$32=2^5$

In general if $n(S) = m$, then, $n\,P(S) = 2^m$

Jniversal Set: When we are studying any branch of mathematics the sets with which ve have to deal, are generally subsets of a bigger set. Such a set is called the **Jniversal set** or the **Universe of Discourse**. At the elementary level when we are tudying arithmetic, we have to deal with *whole numbers* only. At that stage the set of *whole numbers* can be treated as *Universal Set*. At a later stage, when we have to leal with negative numbers also and fractions, the set of the *rational numbers* can be reated as the *Universal Set*.

For illustrating certain concepts of the Set Theory, we sometimes consider small sets (sets having small number of elements) to be universal. This is only an academic artificiality.

Exercise 2.1

1. Write the following sets in set builder notation:

i) $\{1, 2, 3, \ldots, 1000\}$ ii) $\{0, 1, 2, \ldots, 100\}$

iii) $\{0, \pm 1, \pm 2, \ldots, \pm 1000\}$ iv) $\{0, -1, -2, \ldots, -500\}$

v) $\{100, 101, 102, \ldots, 400\}$ vi) $\{-100, -101, -102, \ldots, -500\}$

vii) {Peshawar, Lahore, Karachi, Quetta}

viii) {January, June, July} ix) The set of all odd natural numbers

x) The set of all rational numbers

xi) The set of all real numbers between 1 and 2.

xii) The set of all integers between -100 and 1000

2. Write each of the following sets in the descriptive and tabular forms: -

i) $\{x \mid x \in N \wedge x \leq 10\}$ ii) $\{x \mid x \in N \wedge 4 < x < 12\}$

iii) $\{x \mid x \in Z \wedge -5 < x < 5\}$ iv) $\{x \mid x \in E \wedge 2 < x \leq 4\}$

v) $\{x \mid x \in P \wedge x < 12\}$ vi) $\{x \mid x \in O \wedge 3 < x < 12\}$

vii) $\{x \mid x \in E \wedge 4 \leq x \leq 10\}$ viii) $\{x \mid x \in E \wedge 4 < x < 6\}$

ix) $\{x \mid x \in O \wedge 5 \leq x \leq 7\}$ x) $\{x \mid x \in O \wedge 5 \leq x < 7\}$

xi) $\{x \mid x \in N \wedge x + 4 = 0\}$ xii) $\{x \mid x \in Q \wedge x^2 = 2\}$

xiii) $\{x \mid x \in \mathcal{R} \wedge x = x\}$ xiv) $\{x \mid x \in Q \wedge x = -x\}$

xv) $\{x \mid x \in \mathcal{R} \wedge x \neq x\}$ xvi) $\{x \mid x \in \mathcal{R} \wedge x \notin Q\}$

3. Which of the following sets are finite and which of these are infinite?

i) The set of students of your class.

ii) The set of all schools in Pakistan.

iii) The set of natural numbers between 3 and 10.

iv) The set of rational numbers between 3 and 10.

v) The set of real numbers between 0 and 1.

vi) The set of rationales between 0 and 1

vii) The set of whole numbers between 0 and 1

viii) The set of all leaves of trees in Pakistan.

ix) $P(N)$ x) $P\{a, b, c\}$

xi) $\{1,2,3,4,\ldots\}$ xii) $\{1,2,3,\ldots,100000000\}$

xiii) $\{x \mid x \in \mathbb{R} \wedge x \neq x\}$ xiv) $\{x \mid x \in \mathbb{R} \wedge x^2 = -16\}$

xv) $\{x \mid x \in Q \wedge x^2 = 5\}$ xvi) $\{x \mid x \in Q \wedge 0 \leq x \leq 1\}$

4. Write two proper subsets of each of the following sets: -

i) $\{a, b, c\}$ ii) $\{0, 1\}$ iii) N iv) Z

v) Q vi) $\mathbb{R}$ vii) W viii) $\{x \mid x \in Q \wedge 0 < x \leq 2\}$

5. Is there any set which has no proper sub set? If so name that set.

6. What is the difference between $\{a, b\}$ and $\{\{a,b\}\}$?

7. Which of the following sentences are true and which of them are false?

i) $\{1, 2\} = \{2, 1\}$ ii) $\Phi \subseteq \{\{a\}\}$ iii) $\{a\} \subseteq \{\{a\}\}$

iv) $\{a\} \in \{\{a\}\}$ v) $a \in \{\{a\}\}$ vi) $\Phi \in \{\{a\}\}$

8. What is the number of elements of the power set of each of the following sets?

i) $\{\ \}$ ii) $\{0, 1\}$ iii) $\{1,2,3,4,5,6,7\}$

iv) $\{0,1,2,3,4,5,6,7\}$ v) $\{a, \{b, c\}\}$ vi) $\{\{a,b\},\{b,c,\},\{d,e\}\}$

9. Write down the power set of each of the following sets: -

i) $\{9, 11\}$ ii) $\{+,-,\times,\div\}$ iii) $\{\Phi\}$ iv) $\{a, \{b,c\}\}$

S_1 = The set of odd natural numbers and S_2 = The set of even natural numbers, then S_1 and S_2 are disjoint sets.

The set of arts students and the set of science students of a college are disjoint sets.

Overlapping sets: If the intersection of two sets is non-empty but neither is a subset of the other, the sets are called overlapping sets, e.g., if

$L = \{2,3,4,5,6\}$ and $M = \{5,6,7,8,9,10\}$, then L and M are two overlapping sets.

Complement of a set: The complement of a set A, denoted by A' or A^c relative to the universal set U is the set of all elements of U, which do not belong to A.

Symbolically: $A' = \{x \mid x \in U \wedge x \notin A\}$

For example, if U=N, then $E' = O$ and $O' = E$

Example 1: If U = set of alphabets of English language, C = set of consonants, W = set of vowels, then $C' = W$ and $W' = C$.

Difference of two Sets: The Difference set of two sets A and B denoted by $A-B$ consists of all the elements which belong to A but do not belong to B.

The Difference set of two sets B and A denoted by $B-A$ consists of all the elements, which belong to B but do not belong to A.

Symbolically, $A - B = \{x \mid x \in A \wedge x \notin B\}$ and $B - A = \{x \mid x \in B \wedge x \notin A\}$

Example 2: If $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8,9,10\}$, then

$A - B = \{1,2,3\}$ and $B - A = \{6,7,8,9,10\}$.

Notice that $A-B \neq B-A$.

Note: In view of the definition of complement and difference set it is evident that for any set A, $A' = U - A$

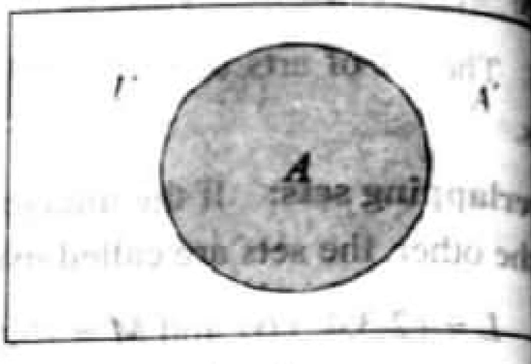
2.3 Venn Diagrams

Venn diagrams are very useful in depicting visually the basic concepts of sets and relationships between sets. They were first used by an English logician and mathematician John Venn (1834 to 1883 A.D).

In a Venn diagram, a rectangular region represents the universal set and regions bounded by simple closed curves represent other sets, which are subsets

universal set. For the sake of beauty these regions are generally shown as circ
regions.

In the adjoining figures, the shaded circular region represents a set A and the remaining portion of rectangle representing the universal set U represents A' or $U-A$.



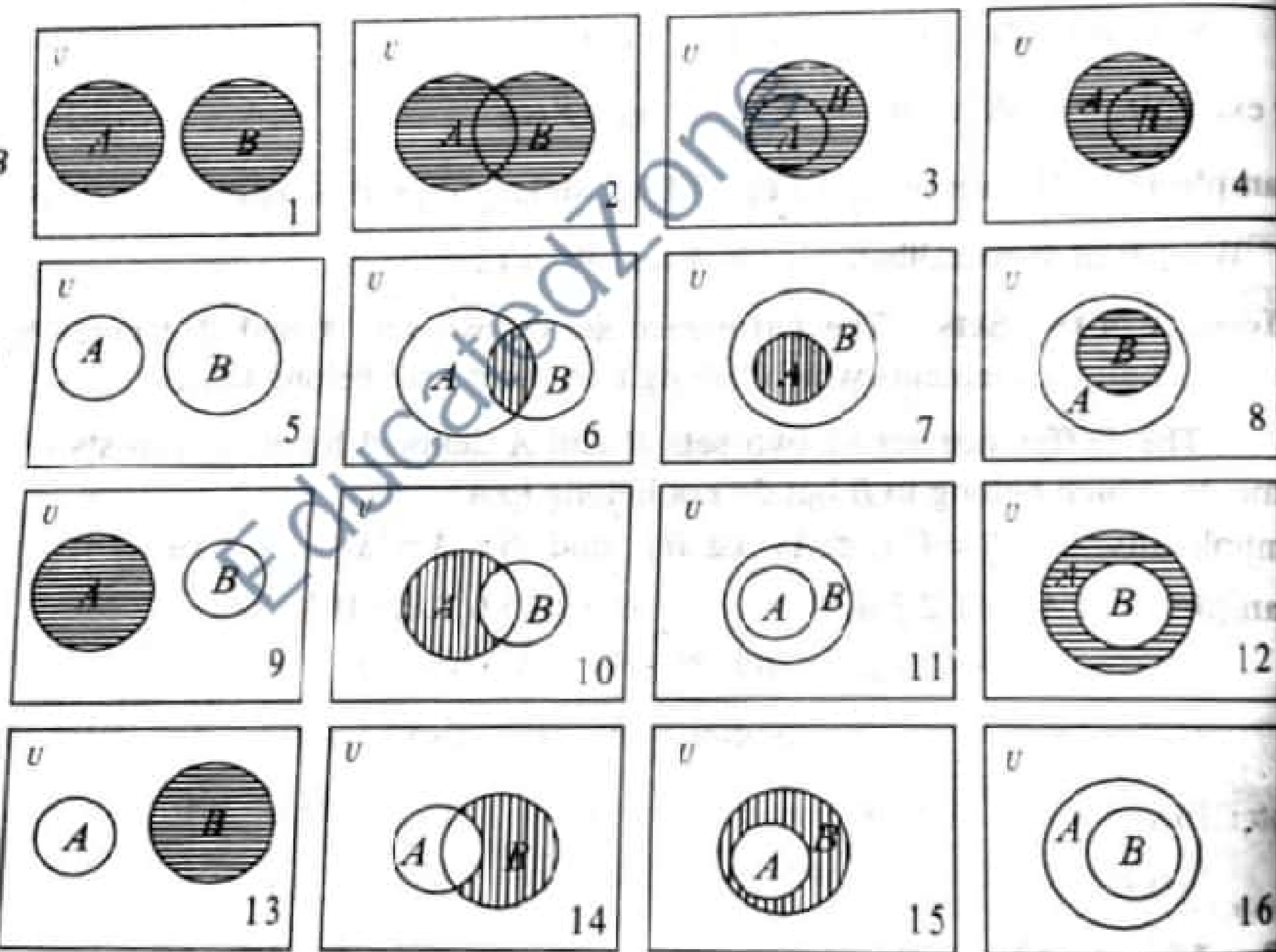
Below are given some more diagrams illustrating basic operations on two sets in different cases (lined region represents result of the relevant operation in each case given below).

$A \cup B$

$A \cap B$

$A - B$

$B - A$



The above diagram suggests the following results: -

Fig No.	Relation between A and B	Result Suggested
1.	A and B disjoint sets $A \cap B = \Phi$	$A \cup B$ consists of all the elements of A and all the elements of B. Also $n(A \cup B) = n(A) + n(B)$
2.	A and B are overlapping $A \cap B \neq \Phi$	$A \cup B$ contains elements which are i) in A and not in B ii) in B and not in A iii) in both A and B. Also $n(A \cup B) = n(A) + n(B) - (A \cap B)$
3.	$A \subseteq B$	$A \cup B = B$; $n(A \cup B) = n(B)$
4.	$B \subseteq A$.	$A \cup B = A$; $n(A \cup B) = n(A)$
5.	$A \cap B = \Phi$	$A \cap B = \Phi$; $n(A \cap B) = 0$
6.	$A \cap B \neq \Phi$	$A \cap B$ contains the elements which are in A and B
7.	$A \subseteq B$	$A \cap B = A$; $n(A \cap B) = n(A)$
8.	$B \subseteq A$	$A \cap B = B$; $n(A \cap B) = n(B)$
9.	A and B are disjoint sets.	$A - B = A$; $n(A - B) = n(A)$
10.	A and B are overlapping	$n(A - B) = n(A) - n(A \cap B)$
11.	$A \subseteq B$	$A - B = \Phi$; $n(A - B) = 0$.
12.	$B \subseteq A$	$A - B \neq \Phi$; $n(A - B) = n(A) - n(B)$
13.	A and B disjoint	$B - A = B$; $n(B - A) = n(B)$
14.	A and B are overlapping	$n(B - A) = n(B) - n(A \cap B)$
15.	$A \subseteq B$	$B - A \neq \Phi$; $n(B - A) = n(B) - n(A)$
16.	$B \subseteq A$	$B - A = \Phi$; $n(B - A) = 0$

Note (1) Since the empty set contains no elements, therefore, no portion of represents it.

(2) If in the diagrams given on preceding page we replace B by the empty (by imagining the region representing B to vanish).

$A \cup \Phi = A$ (From Fig. 1 or 4)

$A \cap \Phi = \Phi$ (From Fig. 5 or 8)

$A - \Phi = A$ (From Fig. 9 or 12)

$\Phi - A = \Phi$ (From Fig. 13 or 16)

Also by replacing B by A (by imagining the regions represented by A a to coincide), we obtain the following results:

$A \cup A = A$ (From fig. 3 or 4)

$A \cap A = A$ (From fig. 7 or 8)

$A - A = \Phi$ (From fig. 12)

Again by replacing B by U, we obtain the results: -

$A \cup U = U$ (From fig. 3); $A \cap U = A$ (From fig. 7)

$A - U = \Phi$ (From fig. 11); $U - A = A'$ (From fig. 15

(3) Venn diagrams are useful only in case of abstract sets whose elements not specified. It is not desirable to use them for concrete sets (Altho this is erroneously done even in some foreign books).

Exercise 2.2

1. Exhibit $A \cup B$ and $A \cap B$ by Venn diagrams in the following cases: -
   i) $A \subseteq B$ ii) $B \subseteq A$ iii) $A \cup A'$
   iv) A and B are disjoint sets. v) A and B are overlapping sets
2. Show $A-B$ and $B-A$ by Venn diagrams when: -
   i) A and B are overlapping sets ii) $A \subseteq B$ iii) $B \subseteq A$
3. Under what conditions on A and B are the following statements true?
   i) $A \cup B = A$ ii) $A \cup B = B$ iii) $A - B = A$
   iv) $A \cap B = B$ v) $n(A \cup B) = n(A) + n(B)$ vi) $n($

vii) $A - B = A$ viii) $n(A \cap B) = 0$ ix) $A \cup B = U$

x) $A \cup B = B \cup A$ xi) $n(A \cap B) = n(B)$. xii) $U - A = \Phi$

4. Let $U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{2,4,6,8,10\}$, $B = \{1,2,3,4,5\}$ and $C = \{1,3,5,7,9\}$

List the members of each of the following sets: -

i) A^c ii) B^c iii) $A \cup B$ iv) $A - B$

v) $A \cap C$ vi) $A^c \cup C^c$ vii) $A^c \cup C$ viii) U^c

5. Using the Venn diagrams, if necessary, find the single sets equal to the following: -

i) A^c ii) $A \cap U$ iii) $A \cup U$ iv) $A \cup \Phi$ v) $\Phi \cap \Phi$

6. Use Venn diagrams to verify the following: -

i) $A - B = A \cap B^c$ ii) $(A - B)^c \cap B = B$

2.4 Operations on Three Sets

If A, B and C are three given sets, operations of union and intersection can be performed on them in the following ways: -

i) $A \cup (B \cup C)$ ii) $(A \cup B) \cup C$ iii) $A \cap (B \cup C)$

iv) $(A \cap B) \cap C$ v) $A \cup (B \cap C)$ vi) $(A \cap C) \cup (B \cap C)$

vii) $(A \cup B) \cap C$ viii) $(A \cap B) \cup C$ ix) $(A \cup C) \cap (B \cup C)$

Let $A = \{1, 2, 3\}$, $B=\{2,3,4,5\}$ and $C = \{3,4,5,6,7,8\}$

We find sets (i) to (iii) for the three sets (Find the remaining sets yourselves).

i) $B \cup C = \{2,3,4,5,6,7,8\}$, $A \cup (B \cup C) = \{1,2,3,4,5,6,7,8\}$

ii) $A \cup B = \{1,2,3,4,5\}$, $(A \cup B) \cup C = \{1,2,3,4,5,6,7,8\}$

iii) $B \cap C = \{3,4,5\}$, $A \cap (B \cap C) = \{3\}$

2.5 Properties of Union and Intersection

We now state the fundamental properties of union and intersection of two or three sets. Formal proofs of the last four are also being given.

Properties:

i) $A \cup B = B \cup A$ (Commutative property of Union)

ii) $A \cap B = B \cap A$ (Commutative property of Intersection)

iii) $A \cup (B \cup C) = (A \cup B) \cup C$ (Associative property of Union)

iv) $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative property of Intersection).

v) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributivity of Union over intersectio

vi) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributivity of intersection over Uni

vii) $(A \cup B)' = A' \cap B'$
viii) $(A \cap B)' = A' \cup B'$ De Morgan's Laws

Proofs of De Morgan's laws and distributive laws:

i) $(A \cup B)' = A' \cap B'$

Let $x \in (A \cup B)'$

$\Rightarrow \quad x \notin A \cup B$

$\Rightarrow \quad x \notin A$ and $A \notin B$

$\Rightarrow \quad x \in A'$ and $x \in B'$

$\Rightarrow \quad x \in A' \cap B'$ (1)

But x is an arbitrary member of $(A \cup B)'$

Therefore, (1) means that $(A \cup B)' \subseteq A' \cap B'$ (2)

Now suppose that $y \in A' \cap B'$

$\Rightarrow \quad y \in A'$ and $y \in B'$

$\Rightarrow \quad y \notin A$ and $y \notin B$

$\Rightarrow \quad y \notin A \cup B$

$\Rightarrow \quad y \in (A \cup B)'$

Thus $A' \cap B' \subseteq (A \cup B)'$ (3)

From (2) and (3) we conclude that

$$(A \cup B)' = A' \cap B'$$

ii) $(A \cap B)' = A' \cup B'$

It may be proved similarly or deducted from (i) by complementation

iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C)$

$\Rightarrow \quad x \in A$ or $x \in B \cap C$

$\Rightarrow \quad$ If $x \in A$ it must belong to $A \cup B$ and $x \in A \cup C$

$\Rightarrow \quad x \in (A \cup B) \cap (A \cup C)$ (1)

Also if $x \in B \cap C$, then $x \in B$ and $x \in C$.

$\Rightarrow \quad x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow \quad x \in (A \cup B) \cap (A \cup C)$

Thus $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Conversely, suppose that

$y \in (A \cup B) \cap (A \cup C)$

There are two cases to consider: -

$y \in A, y \notin A$

In the first case $y \in A \cup (B \cap C)$

If $y \notin A$, it must belong to B as well as C

i.e., $y \in (B \cap C)$

$\therefore y \in A \cup (B \cap C)$

So in either case

$y \in (A \cup B) \cap (A \cup C) \Rightarrow y \in A \cup (B \cap C)$

thus $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ **(3)**

From (2) and (3) it follows that

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

It may be proved similarly or deducted from (iii) by complementation

Verification of the properties:

Example 1: Let $A = \{1,2,3\}$, $B = \{2,3,4,5\}$ and $C = \{3,4,5,6,7,8\}$

i) $A \cup B = \{1,2,3\} \cup \{2,3,4,5\} = \{1,2,3,4,5\}$

$B \cup A = \{2,3,4,5\} \cup \{1,2,3\} = \{2,3,4,5,1\}$

$\therefore A \cup B = B \cup A$

ii) $A \cap B = \{1,2,3\} \cap \{2,3,4,5\} = \{2,3\}$

$B \cap A = \{2,3,4,5\} \cap \{1,2,3\} = \{2,3\}$

$\therefore A \cap B = B \cap A$

(iii) and (iv) Verify yourselves.

(v) $A \cup (B \cap C) = \{1,2,3\} \cup (\{2,3,4,5\} \cap \{3,4,5,6,7,8\})$

$= \{1,2,3\} \cup \{3,4,5\}$

$= \{1,2,3,4,5\}$ **(1)**

$(A \cup B) \cap (A \cup C) = (\{1,2,3\} \cup \{2,3,4,5\}) \cap (\{1,2,3\} \cup \{3,4,5,6,7,8\})$

$= \{1,2,3,4,5\} \cap \{1,2,3,4,5,6,7,8\}$

$= \{1,2,3,4,5\}$ **(2)**

From (1) and (2), $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

vi) Verify yourselves.

vii) Let the universal set be $U = \{1,2,3,4,5,6,7,8,9,10\}$

$A \cup B = \{1,2,3\} \cup \{2,3,4,5\} = \{1,2,3,4,5\}$

$(A \cup B)' = \{6,7,8,9,10\}$ (1

$$A' = U - A = \{4,5,6,7,8,9,10\}$$
$$B' = U - B = \{1,6,7,8,9,10\}$$
$$A' \cap B' = \{4,5,6,7,8,9,10\} \cap \{1,6,7,8,9,10\}$$
$$= \{6,7,8,9,10\} \qquad (2)$$

From (1) and (2), $\quad (A \cup B)' = A' \cap B'$

viii) Verify yourselves.

Verification of the properties with the help of Venn diagrams.

i) and (ii): Verification is very simple, therefore, do it yourselves.

iii): In fig. (1) set A is represented by vertically lined region and $B \cup C$ is represented by horizontally lined region. The set $A \cup (B \cup C)$ is represented by the region which is lined either in one or both ways.

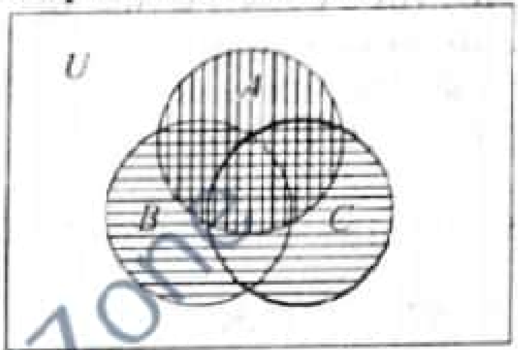


Fig. (1)

In figure(2) $A \cup B$ is represented by horizontally lined region and C by vertically lined region. $(A \cup B) \cup C$ is represented by the region which is lined in either one or both ways.

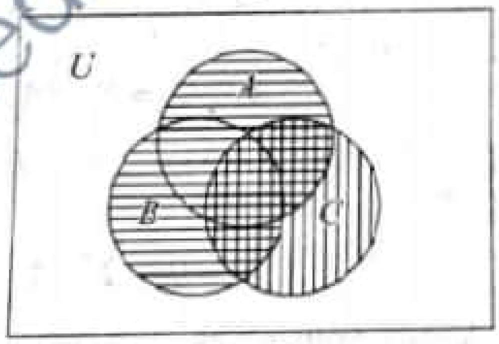


Fig. (2)

From fig (1) and (2) we can see that $A \cup (B \cup C) = (A \cup B) \cup C$

(iv) In fig (3) doubly lined region represents.

$A \cap (B \cap C)$

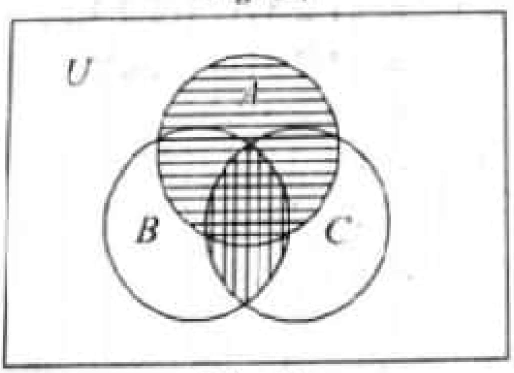


Fig. (3)

(viii) Verify yourselves.

Note: In all the above Venn diagrams only overlapping sets have been considered. Verification in other cases can also be effected similarly. Detail of verification may be written by yourselves.

Exercise 2.3

1. Verify the commutative properties of union and intersection for the following pairs of sets: -
   i) $A = \{1,2,3,4,5\}$, $B = \{4,6,8,10\}$ ii) N, Z
   iii) $A = \{x \mid x \in \mathbb{R} \wedge x \geq 0\}$, $B = \mathbb{R}$.

2. Verify the properties for the sets A, B and C given below: -
   i) Associativity of Union ii) Associativity of intersection.
   iii) Distributivity of Union over intersection.
   iv) Distributivity of intersection over union.
   a) $A = \{1,2,3,4\}$, $B = \{3,4,5,6,7,8\}$, $C = \{5,6,7,9,10\}$
   b) $A = \Phi$, $B = \{0\}$, $C = \{0,1,2\}$
   c) N, Z, Q

3. Verify De Morgan's Laws for the following sets:
   $U = \{1,2,3, \ldots, 20\}$, $A = \{2,4,6, \ldots, 20\}$ and $B=\{1,3,5, \ldots, 19\}$.

4. Let U = The set of the English alphabet
   $A = \{x \mid x \text{ is a vowel}\}$, $B = \{y \mid y \text{ is a consonant}\}$,
   Verify De Morgan's Laws for these sets.

5. With the help of Venn diagrams, verify the two distributive properties in the following cases w.r.t union and intersection.

   i) $A \subseteq B$, $A \cap C = \Phi$ and B and C are overlapping.
   ii) A and B are overlapping, B and C are overlapping but A and C are disjoint.

6 Taking any set, say $A = \{1,2,3,4,5\}$ verify the following: -
   i) $A \cup \Phi = A$ ii) $A \cup A = A$ iii) $A \cap A = A$

7. If $U = \{1,2,3,4,5, \ldots, 20\}$ and $A = \{1,3,5, \ldots, 19\}$, verify the following: -
   i) $A \cup A' = U$ ii) $A \cap U = A$ iii) $A \cap A' = \Phi$

8. From suitable properties of union and intersection deduce the following results:
   i) $A \cap (A \cup B) = A \cup (A \cap B)$ ii) $A \cup (A \cap B) = A \cap (A \cup B)$.

9. Using venn diagrams, verify the following results.
   i) $A \cap B' = A$ iff $A \cap B = \Phi$ ii) $(A - B) \cup B = A \cup B$.
   iii) $(A - B) \cap B = \Phi$ iv) $A \cup B = A \cup (A' \cap B)$.

2.10 Functions

A very important special type of relation is a function defined as below: -

Let A and B be two non-empty sets such that:

i) f is a relation from A to B that is, f is a subset of $A \times B$

ii) Dom $f = A$

iii) First element of no two pairs of f are equal, then f is said to be a function from A to B.

The function f is also written as:

$$f: A \to B$$

which is read: f is a function from A *to* B.

If (x, y) in an element of f when regarded as a set of ordered pairs, we write $y = f(x)$. y is called the value of f for x or image of x under f.

In example 1 discussed above

i) r is a subset of $C \times F$;

ii) Dom $r = \{c_1, c_2, c_3\} = C$;

iii) First elements of no two related pairs of r are the same.

Therefore, r in a function from C to F.

In Example 2 discussed above

i) r is a subset of $A \times A$;

ii) Dom $r \neq A$

Therefore, the relation in this case is not a function.

In example 3 discussed above

i) r is a subset of $\mathbb{R}$.

ii) Dom $r = \mathbb{R}$

iii) Clearly first elements of no two ordered pairs of r can be equal. Therefore, in this case r is a function.

i) Into Function: If a function $f: A \to B$ is such that Ran $f \subset B$ i.e., Ran $f \neq B$, then f is said to be a function from A **into** B. In fig.(1) f is clearly a function. But Ran $f \neq B$. Therefore, f is a function from A into B.

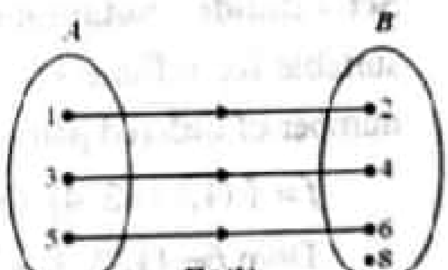


Fig (1)

$$f = \{(1,2), (3,4), (5,6)\}$$

ii) **Onto (Surjective) function:** If a function $f : A \rightarrow B$ is such that Ran $f = B$ i.e., every element of B is the image of some elements of A, then f is called an **onto** function or a surjective function.

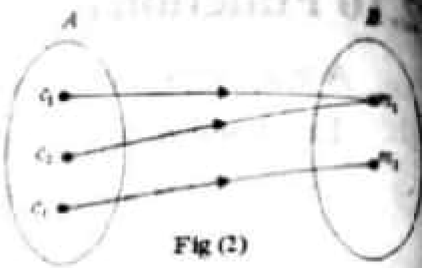


Fig (2)

$f = \{(c_1, m_1), (c_2, m_1), (c_3, m_2)\}$

iii) **(1-1) and into (Injective) function:** If a function f from A into B is such that second elements of no two of its ordered pairs are equal, then it is called an injective (1 – 1, and into) function. The function shown in fig (3) is such a function.

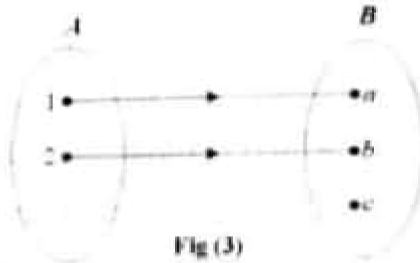


Fig (3)

$f = \{(1, a), (2, b)\}$

iv) **(1 – 1) and Onto function (bijective function).** If f is a function from A onto B such that second elements of no two of its ordered pairs are the same, then f is said to be (1 – 1) function from A onto B.

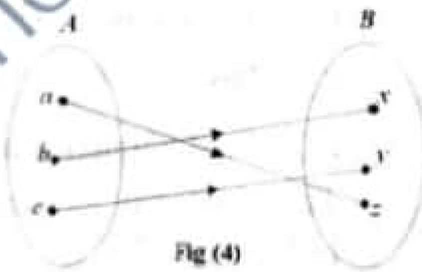


Fig (4)

$f = \{(a, z), (b, x), (c, y)\}$

Such a function is also called a (1–1) correspondence between A and B. It is also called a bijective function. Fig(4) shows a (1–1) correspondence between the sets A and B.

(a, z), (b, x) and (c, y) are the pairs of corresponding elements i.e., in this case $f = \{(a, z), (b, x), (c, y)\}$ which is a bijective function or (1–1) correspondence between the sets A and B.

Set – Builder Notation for a function: We know that sub-builder notation is more suitable for infinite sets. So is the case in respect of a function comprising infinite number of ordered pairs. Consider for instance, the function

$f = \{(1,1), (2, 4), (3, 9), (4, 16), \ldots\}$

Dom $f = \{1, 2, 3, 4, \ldots\}$.and Ran $f = \{1, 4, 9, 16, \ldots\}$

This function may be written as: $f = \{(x, y) \mid y = x^2, x \in N\}$

For the sake of brevity this function may be written as:

f = function defined by the equation $y = x^2$, $x \in N$.

Or, to be still more brief: The function x^2, $x \in N$.

In algebra and Calculus the domain of most functions is $\mathcal{R}$ and if evident from the context it is, generally, omitted.

2.10.1 Linear and Quadratic Functions

The function $\{(x, y) \mid y = mx + c\}$ is called a **linear function**, because its graph (geometric representation) is a straight line. Detailed study of a straight line will be undertaken in the next class. For the present it is sufficient to know that an equation of the form

$y = mx + c$ or $ax + by + c = 0$ represents a straight line. This can be easily verified by drawing graphs of a few linear equations with numerical coefficients. The function $\{(x, y) \mid y = ax^2 + bx + c\}$ is called a **quadratic function** because it is defined by a quadratic (second degree) equation in x, y.

Example 4: Give rough sketch of the functions

i) $\{(x, y) \mid 3x + y = 2\}$ ii) $\{(x, y) \mid y = \frac{1}{2}x^2\}$

Solution:

i) The equation defining the function is $3x + y = 2$

$\Rightarrow \quad y = -3x + 2$

We know that this equation, being linear, represents a straight line. Therefore, for drawing its sketch or graph only two of its points are sufficient.

When $x = 0$, $y = 2$,

When $y = 0$, $x = \frac{2}{3} = 0.6$ nearly. So two points on the line are $A\ (0, 2)$ and $B = (0.6, 0)$.

Joining A and B and producing $\overline{AB}$ in both directions, we obtain the line AB i.e., graph of the given function.

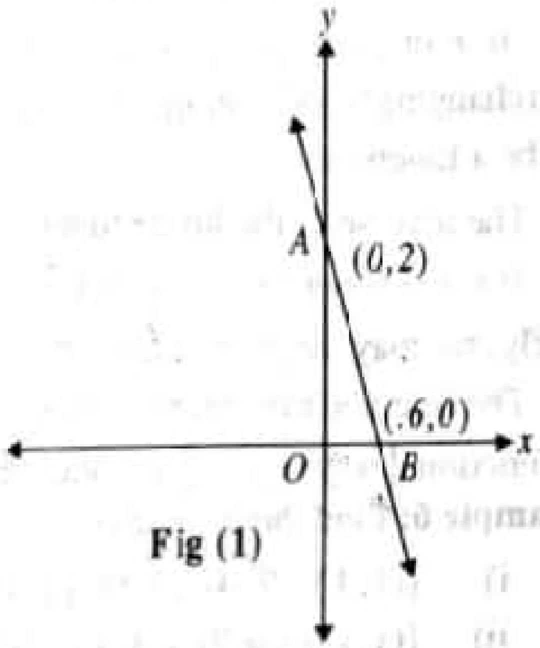


Fig (1)

ii) The equation defining the function is $y = \frac{1}{2}x^2$.

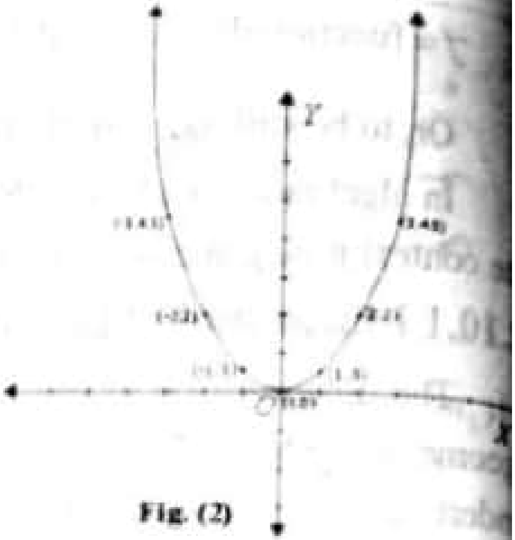


Fig. (2)

orresponding to the values 0, $\pm$ 1, $\pm$2, $\pm$3 ...
lues of y are 0, .5, 2, 4.5, ...

We plot the points (0, 0), ($\pm$ 1, .5), ($\pm$ 2, 2), ($\pm$ 3, 4.5), ... Joining them by means of a smooth curve and extending it upwards we get the required graph. We notice that:

i) The entire graph lies above the x-axis.

ii) Two equal and opposite values of x correspond to every value of y (but not vice versa).

iii) As x increases (numerically) y increases and there is no end to their increase. Thus the graph goes infinitely upwards. Such a curve is called a parabola. The students will learn more about it in the next class.

2.11 Inverse of a function

If a relation or a function is given in the tabular form i.e., as a set of ordered pairs, its inverse is obtained by interchanging the components of each ordered pair. The inverse of r and f are denoted r^{-1} and f^{-1} respectively.

If r or f are given in set-builder notation the inverse of each is obtained by interchanging x and y in the defining equation. The inverse of a function may or may not be a function.

The inverse of the linear function

$\{(x, y) \mid y = mx + c\}$ is $\{(x, y) \mid x = my + c\}$ which is also a linear function. Briefly, we may say that the **inverse of a line is a line.**

The line $y = x$ is clearly self-inverse. The function defined by this equation i.e., the function $\{(x, y) \mid y = x\}$ is called the identity function.

Example 6: Find the inverse of

i) $\{(1, 1), (2, 4), (3, 9), (4, 16), \dots\ x \in Z^+\}$,

ii) $\{(x, y) \mid y = 2x + 3, x \in \mathcal{R}\}$ iii) $\{(x, y) \mid x^2 + y^2 = a^2\}$.

Tell which of these are functions.

Solution:

i) The inverse is:

$\{(2, 1), (4, 2), (9, 3), (16, 4) \ldots\}$.

This is also a function.

Note: Remember that the equation

$y = \sqrt{x}, x \geq 0$

defines a function but the equation $y^2 = x, x \geq 0$ does not define a function.

The function defined by the equation

$$y = \sqrt{x},\ x \geq 0$$

is called the **square root function.**

The equation $y^2 = x \Rightarrow y = \pm\sqrt{x}$

Therefore, the equation $y^2 = x\ (x \geq 0)$ may be regarded as defining the union of the functions defined by

$$y = \sqrt{x}, x \geq 0 \text{ and } y = -\sqrt{x}, x \geq 0.$$

i) The given function is a **linear function.** Its inverse is:

$\{(x, y) \mid x = 2y + 3\}$

which is also a linear function.

Points $(0, 3), (-1.5, 0)$ lie on the given line and points $(3, 0), (0, -1.5)$ lie on its nverse. (Draw the graphs yourselves).

The lines l, l' are symmetric with respect to the line $y = x$. This quality of ymmetry is true not only about a linear n function and its inverse but is also true bout any function of a higher degree and its inverse (why?).

Exercise 2.6

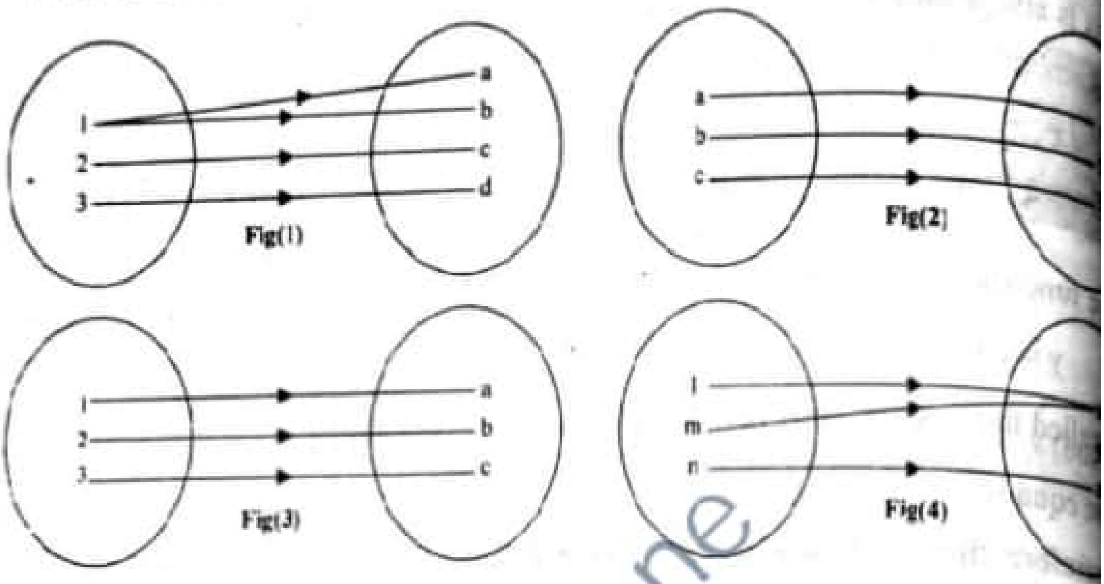
• For $A = \{1, 2, 3, 4\}$, find the following relations in A. State the domain and range of each relation. Also draw the graph of each.

i) $\{(x, y) \mid y = x\}$ ii) $\{(x, y) \mid y + x = 5\}$

iii) $\{(x, y) \mid x + y < 5\}$ iv) $\{(x, y) \mid x + y > 5\}$

2. Repeat Q-1 when $A = \mathcal{R}$, the set of real numbers. Which of the real li functions.

3. Which of the following diagrams represent functions and of which type?



4. Find the inverse of each of the following relations. Tell whether each and its inverse is a function or not: -

i) $\{(2,1), (3,2), (4,3), (5,4), (6,5)\}$ ii) $\{(1,3), (2,5), (3,7), (4,9), (5,11)\}$

iii) $\{(x, y) \mid y = 2x+3, x \in \mathcal{R}\}$ iv) $\{(x, y) \mid y^2 = 4ax, x \geq 0\}$

v) $\{(x, y) \mid x^2 + y^2 = 9, \mid x \mid \leq 3, \mid y \mid \leq 3\}$

2.12 Binary Operations

In lower classes we have been studying different number systems inve the properties of the operations performed on each system. Now we a opposite course. We now study certain operations which may be useful in particular cases.

An operation which when performed on a single number yields another of the same or a different system is called a **unary operation**.

Examples of *Unary operations* are negation of a given number, extra square roots or cube roots of a number, squaring a number or raising it to a power.

We now consider binary operation, of much greater importance, which requires two numbers. We start by giving a formal definitio operation.

A *binary operation* denoted as ※ (read as star) on a non-empty set G is a function which associates with each ordered pair (a, b), of elements of G, a unique element, denoted as $a ※ b$ of G.

In other words, a binary operation on a set G is a function from the set $G \times G$ to the set G. For convenience we often omit the word *binary* before *operation*.

Also in place of saying ※ is an operation on G, we shall say G is closed with respect to ※.

Example 1: Ordinary addition, multiplication are operations on N. i.e., N is closed with respect to ordinary addition and multiplication because

$$\forall a, b \in N,\ a+b \in N \wedge a.b \in N$$

($\forall$ stands for "for all" and $\wedge$ stands for "and")

Example 2: Ordinary addition and multiplication are operations on E, the set of all even natural numbers. It is worth noting that addition is not an operation on O, the set of old natural numbers.

Example 3: With obvious modification of the meanings of the symbols, let E be any even natural number and O be any odd natural number, then

$E \oplus E = E$ (Sum of two even numbers is an even number).

$E \oplus O = O$

and $O \oplus O = E$

$\oplus$	E	O
E	E	O
O	O	E

These results can be beautifully shown in the form of a table given above:

This shows that the set $\{E, O\}$ is closed under (ordinary) addition.

The table may be read (horizontally).

$E \oplus E = E$, $\qquad E + O = O$;

$O \oplus O = E$, $\qquad O + E = O$

Example 4: The set $\{1, -1, i, -i\}$ where $i = \sqrt{-1}$ is closed w.r.t multiplication (but not $w.\ r.\ t$ addition). This can be verified from the adjoining table.

$\otimes$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Note: The elements of the set of this example are the fourth roots of unity.

Example 5: It can be easily verified that ordinary multiplication (but not addition) is an operation on the set $\{1, \omega, \omega^2\}$ where $\omega^3 = 1$. The adjoining table may be used for the verification of this fact.

(ω is pronounced omega)

$\otimes$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω