4 Quadratic Equations

4.1 Introduction

A *quadratic equation* in x is an equation that can be written in the form $ax^2+bx+c=0$; where a, b and c are real numbers and $a \neq 0$.

Another name for a quadratic equation in x is **2nd Degree Polynomial** in x.

The following equations are the quadratic equations:

i)	$x^2-7x+10=0$;	$a=1,\ b=-7,$	$c=10$
ii)	$6x^2+x-15=0$;	$a=6,\ b=1,$	$c=-15$
iii)	$4x^2+5x+3=0$;	$a=4,\ b=5,$	$c=3$
iv)	$3x^2-x=0$;	$a=3,\ b=-1,$	$c=0$
v)	$x^2=4$;	$a=1,\ b=0,$	$c=-4$

4.1.1 Solution of Quadratic Equations

There are three basic techniques for solving a quadratic equation:

i) by factorization.

ii) by completing squares, extracting square roots.

iii) by applying the quadratic formula.

By Factorization: It involves factoring the polynomial ax^2+bx+c.

It makes use of the fact that if $ab=0$, then $a=0$ or $b=0$.

For example, if $(x-2)(x-4)=0$, then either $x-2=0$ or $x-4=0$.

Example 1: Solve the equation $x^2-7x+10=0$ by factorization.

Solution:

$$x^2-7x+10=0$$

$$\Rightarrow \quad (x-2)(x-5)=0$$

$\therefore$ either $x-2=0 \Rightarrow x=2$

or $x-5=0 \Rightarrow x=5$

$\therefore$ the given equation has two solutions: 2 and 5

$\therefore$ solution set $=\{2,5\}$

Note: The solutions of an equation are also called its roots.

$\therefore$ 2 and 5 are roots of $x^2-7x+10=0$

By Completing Squares, then Extracting Square Roots: Sometimes, the quadratic polynomials are not easily factorable.

For example, consider $x^2 + 4x - 437 = 0$.

It is difficult to make factors of $x^2 + 4x - 437$. In such a case the factorization and hence the solution of quadratic equation can be found by the method of completing the square and extracting square roots.

Example 2: Solve the equation $x^2 + 4x - 437 = 0$ by completing the squares.

Solution : $x^2 + 4x - 437 = 0$

$$\Rightarrow \quad x^2 + 2\left(\frac{4}{2}\right)x = 437$$

Add $\left(\frac{4}{2}\right)^2 = (2)^2$ to both sides

$$x^2 + 4x + (2)^2 = 437 + (2)^2$$

$$\Rightarrow \quad (x+2)^2 = 441$$

$$\Rightarrow \quad x + 2 = \pm\sqrt{441} = \pm 21$$

$$\Rightarrow \quad x = \pm 21 - 2$$

$$\therefore \quad x = 19 \text{ or } x = -23$$

Hence solution set $= \{-23, 19\}$.

By Applying the Quadratic Formula: Again there are some quadratic polynomials which are not factorable at all using integral coefficients. In such a case we can always find the solution of a quadratic equation $ax^2 + bx + c = 0$ by applying a formula known as quadratic formula. This formula is applicable for every quadratic equation.

Derivation of the Quadratic Formula

Standard form of quadratic equation is

$$ax^2 + bx + c = 0, \ a \neq 0$$

Step 1. Divide the equation by a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Step 2. Take constant term to the R.H.S.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Step 3. To complete the square on the L.H.S. add $\left(\frac{b}{2a}\right)^2$ to both sides.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence the solution of the quadratic equation $ax^2 + bx + c = 0$ is given by

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

which is called **Quadratic Formula**.

Example 3: Solve the equation $6x^2 + x - 15 = 0$ by using the quadratic formula.

Solution: Comparing the given equation with $ax^2 + bx + c = 0$, we get,

$a = 6,\ b = 1, c = -15$

$\therefore$ The solution is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1^2 - 4(6)(-15)}}{2(6)}$$

$$= \frac{-1 \pm \sqrt{361}}{12} = \frac{-1 \pm 19}{12}$$

i.e., $x = \dfrac{-1+19}{12}$ or $x = \dfrac{-1-19}{12}$

$x = \dfrac{3}{2}$ or $x = \dfrac{-5}{3}$ Hence solution set $= \left\{\dfrac{3}{2}, \dfrac{-5}{3}\right\}$

Example 4: Solve the $8x^2 - 14x - 15 = 0$ by using the quadratic formula.

Solution: Comparing the given equation with $ax^2 + bx + c = 0$, we get,

$a = 8,\ b = -14,\ c = -15$

$$\therefore \quad x = \frac{-(-14) \pm \sqrt{(-14)\ldots}}{2(8)}$$

$$= \frac{14 \pm \sqrt{676}}{16} = \frac{14 \pm 26}{16}$$

$$\therefore \text{ either } x = \frac{14 + 26}{16} \Rightarrow x = \frac{5}{2}$$

$$\text{or } x = \frac{14 - 26}{16} \Rightarrow x = \frac{-3}{4}$$

Hence Solution set $= \left\{\frac{5}{2}, -\frac{3}{4}\right\}$

Exercise 4.1

Solve the following equations by factorization:

1. $3x^2 + 4x + 1 = 0$
2. $x^2 + 7x + 12 = 0$
3. $9x^2 - 12x - 5 = 0$
4. $x^2 - x = 2$
5. $x(x + 7) = (2x - 1)(x + 4)$
6. $\frac{x}{x+1} + \frac{x+1}{x} = \frac{5}{2}; x \neq -1, 0$
7. $\frac{1}{x+1} + \frac{2}{x+2} = \frac{7}{x+5}; x \neq -1, -2, -5$
8. $\frac{a}{ax-1} + \frac{b}{bx-1} = a + b; x \neq \frac{1}{a}, \frac{1}{b}$

Solve the following equations by completing the square:

9. $x^2 - 2x - 899 = 0$
10. $x^2 + 4x - 1085 = 0$
11. $x^2 + 6x - 567 = 0$
12. $x^2 - 3x - 648 = 0$
13. $x^2 - x - 1806 = 0$
14. $2x^2 + 12x - 110 = 0$

Find roots of the following equations by using quadratic formula:

15. $5x^2 - 13x + 6 = 0$
16. $4x^2 + 7x - 1 = 0$

17. $15x^2 + 2ax - a^2 = 0$
18. $16x^2 + 8x + 1 = 0$
19. $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$
20. $(a+b)x^2 + (a+2b+c)x + b + c = 0$

4.2 Solution of Equations Reducible to the Quadratic Equation

There are certain types of equations, which do not look to be of degree 2, but they can be reduced to the quadratic form. We shall discuss the solutions of such five types of the equations one by one.

Type I: The equations of the form: $ax^{2n} + bx^{n} + c = 0, \; a \neq 0$

Put $x^n = y$ and get the given equation reduced to quadratic equation in y.

Example 1: Solve the equation: $x^{\frac{1}{2}} - x^{\frac{1}{4}} - 6 = 0.$

Solution This given equation can be written as $(x^{\frac{1}{4}})^2 - x^{\frac{1}{4}} - 6 = 0$

Let $x^{\frac{1}{4}} = y$

$\therefore$ The given equation becomes

$$y^2 - y - 6 = 0$$

$\Rightarrow (y-3)(y+2) = 0$

$\Rightarrow y = 3,$ or $y = -2$

$\therefore x^{\frac{1}{4}} = 3$ | $x^{\frac{1}{4}} = -2$

$\Rightarrow x = (3)^4$ | $\Rightarrow x = (-2)^4$

$\Rightarrow x = 81$ | $\Rightarrow x = 16$

Hence solution set is $\{16, 81\}$.

Type II: **The equation of the form:** $(x+a)(x+b)(x+c)(x+d) = k$

where $a + b = c + d$

Example 2: Solve $(x-7)(x-3)(x+1)(x+5) - 1680 = 0$

Solution : $(x-7)(x-3)(x+1)(x+5) - 1680 = 0$

$\Rightarrow [(x-7)(x+5)][(x-3)(x+1)] - 1680 = 0$ (by grouping)

$\Rightarrow (x^2 - 2x - 35)(x^2 - 2x - 3) - 1680 = 0$

Solution : $(x-7)(x-3)(x+1)(x+5)-1680 \quad =0$

$\Rightarrow \quad [(x-7)(x+5)][(x-3)(x+1)]-1680 = 0$ (by grouping)

$\Rightarrow \quad (x^2-2x-35)(x^2-2x-3)-1680=0$

Putting $x^2-2x=y$, the above equation becomes

$(y-35)(y-3)-1680=0$

$\Rightarrow \quad y^2-38y+105-1680=0$

$\Rightarrow \quad y^2-38y-1575=0$

$\therefore y=\dfrac{38\pm\sqrt{1444+6300}}{2}=\dfrac{38\pm\sqrt{7744}}{2}$ (by quadratic formula)

$=\dfrac{38\pm 88}{2}$

$\Rightarrow y=63$

$\Rightarrow x^2-2x=63$

$\Rightarrow x^2-2x-63 \quad =0$

$\Rightarrow (x+7)(x-9) \quad =0$

$\Rightarrow x=-7 \text{ or } x=9$

or $\quad y=-25.$

$\Rightarrow x^2-2x=-25$

$\Rightarrow x^2-2x+25=0$

$\Rightarrow x=\dfrac{2\pm\sqrt{4-100}}{2}$

$=\dfrac{2\pm\sqrt{-96}}{2}$

$=\dfrac{2\pm 4\sqrt{6}\,i}{2} = 1\pm 2\sqrt{6}\,i$

$\Rightarrow x=1+2\sqrt{6}\,i \text{ or } x=1-2\sqrt{6}\,i$

Hence Solution set $=\{-7, 9, 1+2\sqrt{6}\,i, 1-2\sqrt{6}\,i\}$

Type III: Exponential Equations: Equations, in which the variable occurs in exponent, are called **exponential equations**. The method of solving such equations is explained by the following examples.

Example 3: Solve the equation: $2^{2x}-3.2^{x+2}+32=0$

Solution : $2^{2x}-3.2^{x+2}+32 \quad =0$

$\Rightarrow \quad 2^{2x}-3.2^2.2^x+32=0$

$\Rightarrow \quad 2^{2x}-12.2^x+32 \quad =0$

$\Rightarrow \quad y^2-12y+32 \quad =0$ (Putting $2^x=y$)

$\Rightarrow \quad (y-8)(y-4) \quad =0$

$\Rightarrow y = 8$ or $y = 4$

$\Rightarrow 2^x = 8$

$\Rightarrow 2^x = 2^3$

$\Rightarrow x = 3$

$\Rightarrow 2^x = 4$

$\Rightarrow 2^x = 2^2$

$\Rightarrow x = 2$

Hence solution set $= \{2, 3\}$.

Example 4: Solve the equation: $4^{1+x} + 4^{1-x} = 10$

Solution: Given that

$+ 4^{1-x} = 10$

$+ 4.4^{-x} = 10$

Let $4^x = y \Rightarrow 4^{-x} = (4^x)^{-1} = y^{-1} = \frac{1}{y}$

$\therefore$ The given equation becomes

$$4y + \frac{4}{y} - 10 = 0$$

$$\Rightarrow 4y^2 - 10y + 4 = 0$$

$$\Rightarrow 2y^2 - 5y + 2 = 0$$

$$\therefore y = \frac{5 \pm \sqrt{25 - 4(2)(2)}}{2(2)} = \frac{5 \pm \sqrt{9}}{4} = \frac{5 \pm 3}{4}$$

$\Rightarrow y = 2$ or $y = \frac{1}{2}$

$\therefore 4^x = 2$

$\Rightarrow 2^{2x} = 2^1$

$\Rightarrow 2x = 1$

$\Rightarrow x = \frac{1}{2}$

$\therefore 4^x = \frac{1}{2}$

$\Rightarrow 2^{2x} = 2^{-1}$

$\Rightarrow 2x = -1$

$\Rightarrow x = -\frac{1}{2}$

Hence Solution set $= \left\{\frac{1}{2}, -\frac{1}{2}\right\}$.

Type IV: Reciprocal Equations: An equation, which remains unchanged when x is replaced by $\frac{1}{x}$, is called a reciprocal equation. In such an equation the coefficien

$\Rightarrow \quad y = 2$ or $y = \frac{1}{2}$

$\therefore \quad 4^x = 2$

$\Rightarrow \quad 2^{2x} = 2^1$

$\Rightarrow \quad 2x = 1$

$\Rightarrow \quad x = \frac{1}{2}$

$\therefore \quad 4^x = \frac{1}{2}$

$\Rightarrow \quad 2^{2x} = 2^{-1}$

$\Rightarrow \quad 2x = -1$

$\Rightarrow \quad x = -\frac{1}{2}$

Hence Solution set $= \left\{\frac{1}{2}, -\frac{1}{2}\right\}$.

Type IV: Reciprocal Equations: An equation, which remains unchanged when x is replaced by $\frac{1}{x}$, is called a reciprocal equation. In such an equation the coefficients of

the terms equidistant from the beginning and end are equal in magnitude. The method of solving such equations is explained through the following example:

Example 5: Solve the equation

$$x^4 - 3x^3 + 4x^2 - 3x + 1 = 0;$$

Solution: Given that:

$$x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$$

$$\Rightarrow \quad x^2 - 3x + 4 - \frac{3}{x} + \frac{1}{x^2} = 0 \qquad \text{(Dividing by } x^2\text{)}$$

$$\Rightarrow \quad \left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) + 4 = 0 \qquad (1)$$

Let $x + \frac{1}{x} = y \quad \Rightarrow x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = y^2 - 2$

So, the equation (1) reduces to

$y^2 - 2 - 3y + 4 = 0$

$\Rightarrow \quad y^2 - 3y + 2 = 0$

$\Rightarrow \quad (y - 2)(y - 1) = 0$

$\Rightarrow \quad y = 2$ or $y = 1$

$\Rightarrow \quad x + \frac{1}{x} = 2$

$\Rightarrow \quad x^2 - 2x + 1 = 0$

$\Rightarrow \quad (x - 1)^2 = 0$

$\Rightarrow \quad (x - 1)(x - 1) = 0$

$\Rightarrow \quad x = 1, 1$

$\Rightarrow \quad x + \frac{1}{x} = 1$

$\Rightarrow \quad x^2 - x + 1 = 0$

$\Rightarrow \quad x = \frac{1 \pm \sqrt{1 - 4}}{2}$

$\Rightarrow \quad x = \frac{1 \pm \sqrt{-3}}{2}$

Hence Solution set $= \left\{1, \frac{1 \pm \sqrt{-3}}{2}\right\}$

3. $x^6 - 9x^3 + 8 = 0$

4. $8x^6 - 19x^3 - 27 = 0$

5. $x^{\frac{2}{5}} + 8 = 6x^{\frac{1}{5}}$

6. $(x+1)(x+2)(x+3)(x+4) = 24$

7. $(x-1)(x+5)(x+8)(x+2) - 880 = 0$

8. $(x-5)(x-7)(x+6)(x+4) - 504 = 0$

9. $(x-1)(x-2)(x-8)(x+5) + 360 = 0$

10. $(x+1)(2x+3)(2x+5)(x+3) = 945$

Hint: $(x+1)(2x+5)(2x+3)(x+3) = 945$

11. $(2x-7)(x^2-9)(2x+5) - 91 = 0$

12. $(x^2 + 6x + 8)(x^2 + 14x + 48) = 105$

13. $(x^2 + 6x - 27)(x^2 - 2x - 35) = 385$

14. $4 \cdot 2^{2x+1} - 9 \cdot 2^x + 1 = 0$

15. $2^x + 2^{-x+6} - 20 = 0$

16. $4^x - 3 \cdot 2^{x+3} + 128 = 0$

17. $\ldots^{2x-1} - 12 \cdot 3^x + 81 = 0$

18. $\left(x + \frac{1}{x}\right)^2 - 3\left(x + \frac{1}{x}\right) - 4 = 0$

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19. $\ldots + \frac{1}{x} + \frac{1}{x^2} = 0$

20. $\left(x - \frac{1}{x}\right)^2 + 3\left(x + \frac{1}{x}\right) = 0$

21. $2x^4 - 3x^3 - x^2 - 3x + 2 = 0$

22. $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

23. $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

24. $x^4 - 6x^2 + 10 - \frac{6}{x^2} + \frac{1}{x^4} = 0$

Type V: Radical Equations: Equations involving **radical expressions** of the variable are called radical equations. To solve a radical equation, we first obtain an equation free from radicals. Every solution of radical equation is also a solution of the radical-free equation but the new equation have solutions that are not solutions of the original radical equation.

Such extra solutions (roots) are called **extraneous roots**. The method of the solution of different types of radical equations is illustrated by means of the followings examples:

i) **The Equations of the form:** $l(ax^2 + bx) + m\sqrt{ax^2 + bx + c} = 0$

Example 1: Solve the equation

$$3x^2 + 15x - 2\sqrt{x^2 + 5x + 1} = 2$$

Solution : Let $\sqrt{x^2 + 5x + 1} = y$

$$\Rightarrow x^2 + 5x + 1 = y^2$$

$$\Rightarrow x^2 + 5x = y^2 - 1$$

$$\Rightarrow 3x^2 + 15x = 3y^2 - 3$$

$\therefore$ The given equation becomes $3y^2 - 3 - 2y = 2$

$$\Rightarrow 3y^2 - 2y - 5 = 0$$

$$\Rightarrow (3y-5)(y+1) = 0$$

5. $\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$

4. $\sqrt{3x+4}=2+\sqrt{2x-4}$

6. $\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1$

7. $\sqrt{x^2+2x-3}+\sqrt{x^2+7x-8}=\sqrt{5(x^2+3x-4)}$

8. $\sqrt{2x^2-5x-3}+3\sqrt{2x+1}=\sqrt{2x^2+25x+12}$

9. $\sqrt{3x^2-5x+2}+\sqrt{6x^2-11x+5}=\sqrt{5x^2-9x+4}$

10. $(x+4)(x+1)=\sqrt{x^2+2x-15}+3x+31$

11. $\sqrt{3x^2-2x+9}+\sqrt{3x^2-2x-4}=13$

12. $\sqrt{5x^2+7x+2}-\sqrt{4x^2+7x+18}=x-4$

4.3 Three Cube Roots of Unity

Let x be a cube root of unity

$\therefore \quad x = \sqrt[3]{1} = (1)^{\frac{1}{3}}$

$\Rightarrow \quad x^3 = 1$

$\Rightarrow \quad x^3 - 1 = 0$

$\Rightarrow \quad (x-1)(x^2+x+1) = 0$

Either $x-1 = 0 \Rightarrow x = 1$

or $\quad x^2+x+1 = 0$

$$\therefore \quad x = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm\sqrt{-3}}{2}$$

$$\Rightarrow \quad x = \frac{-1\pm\sqrt{3}\,i}{2} \quad (\because \sqrt{-1}=i)$$

Thus the three cube roots of unity are:

$$1, \frac{-1+\sqrt{3}\,i}{2} \text{ and } \frac{-1-\sqrt{3}\,i}{2}$$

Note: We know that the numbers containing i are called complex numbers. So $\frac{-1+\sqrt{3}\,i}{2}$ and $\frac{-1-\sqrt{3}\,i}{2}$ are called complex* or imaginary cube roots of unity.

4.8 Relations Between the Roots and the Coefficients of a Quadratic Equation

Let α, β be the roots of $ax^2 + bx + c = 0, a \neq 0$ such that

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\therefore \alpha + \beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a}$$

and
$$\alpha\beta = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}$$

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$$= \frac{b^2-b^2+4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Sum of the roots $= S = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Product of the roots $= P = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$

The above results are helpful in expressing symmetric functions of the roots in terms of the coefficients of the quadratic equations.

Example 1: If α, β are the roots of $ax^2 + bx + c = 0,\ a \neq 0$, find the values of

i) $\alpha^2 + \beta^2$ ii) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ iii) $(\alpha - \beta)^2$

Solution: Since α, β are the roots of $ax^2 + bx + c = 0$

$$\therefore \alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

i)
$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$

$$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2-2ca}{a^2}$$

ii)
$$\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta} = \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta}$$

$$= \frac{\left(-\frac{b}{a}\right)^3 - 3\frac{c}{a}\left(-\frac{b}{a}\right)}{\frac{c}{a}} = \frac{\frac{-b^3+3abc}{a^3}}{\frac{c}{a}}$$

$$= \frac{-b^3 + 3abc}{a^2c}$$

iii) $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$

$$= \left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right) = \frac{b^2}{a^2} - 4\frac{c}{a} = \frac{b^2-4ac}{a^2}$$

Example 2: Find the condition that one root of $ax^2 + bx + c = 0,\ a \neq 0$ is square of the other.

Solution: As one root of $ax^2 + bx + c = 0$ is square of the other,

let the roots be α and α^2

Sum of roots $= \alpha + \alpha^2 = -\frac{b}{a}$ (i)

Product of roots $= \alpha \cdot \alpha^2 = \frac{c}{a} \Rightarrow \alpha^3 = \frac{c}{a}$ (ii)

Cubing both sides of (i), we get

$$\alpha^3 + \alpha^6 + 3\alpha\alpha^2(\alpha + \alpha^2) = -\frac{b^3}{a^3}$$

$$\Rightarrow \alpha^3 + (\alpha^3)^2 + 3\alpha^3(\alpha + \alpha^2) = -\frac{b^3}{a^3}$$

$$\Rightarrow \frac{c}{a} + \left(\frac{c}{a}\right)^2 + 3\frac{c}{a}\left(-\frac{b}{a}\right) = -\frac{b^3}{a^3} \quad \text{(From (i), (ii))}$$

$$\Rightarrow a^2c + ac^2 - 3abc = -b^3$$

9 Formation of an Equation Whose Roots are Given

$\therefore$ $(x-\alpha)(x-\beta) = 0$ has the roots α and β

$\Rightarrow$ $x^2 - (\alpha+\beta)x + \alpha\beta = 0$ has the roots α and β.

For S = Sum of the roots and P = Product of the roots.

Thus $x^2 - Sx + P = 0$

xample 3: If α, β are the roots of $ax^2 + bx + c = 0$, form the equation whose roots e double the roots of this equation.

lution: $\because$ α and β are the roots of $ax^2 + bx + c = 0$

$\therefore$ $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

The new roots are 2α and 2β.

$\therefore$ Sum of new roots $= 2\alpha + 2\beta$

$$= 2(\alpha + \beta) = -\frac{2b}{a}$$

Product of new roots $= 2\alpha \, . \, 2\beta = 4\alpha\beta = \frac{4c}{a}$

Required equation is given by

$y^2 -$ (Sum of roots) $y +$ Product of roots $=0$

$$\Rightarrow \quad y^2 + \frac{2b}{a} y + \frac{4c}{a} = 0 \qquad \Rightarrow \qquad ay^2 + 2by + 4c = 0$$

Exercise 4.6

1. If α, β are the roots of $3x^2 - 2x + 4 = 0$, find the values of

   i) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ ii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ iii) $\alpha^4 + \beta^4$

   iv) $\alpha^3 + \beta^3$ v) $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$ vi) $\alpha^2 - \beta^2$.

2. If α, β are the roots of $x^2 - px - p - c = 0$, prove that

$$(1 + \alpha)(1 + \beta) = 1 - c$$

3. Find the condition that one root of $x^2 + px + q = 0$ is

   i) double the other ii) square of the other

   iii) additive inverse of the other iv) multiplicative inverse of the other.

4. If the roots of the equation $x^2 - px + q = 0$ differ by unity, prove that $p^2 = 4q + 1$.

5. Find the condition that $\frac{a}{x-a} + \frac{b}{x-b} = 5$ may have roots equal in magnitude but opposite in signs.

6. If the roots of $px^2 + qx + q = 0$ are α and β then prove that

$$\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}} = 0.$$

7. If α, β are the roots of the equation $ax^2 + bx + c = 0$, form the equations whose roots are

   i) α^2, β^2 ii) $\frac{1}{\alpha}, \frac{1}{\beta}$ iii) $\frac{1}{\alpha^2}, \frac{1}{}$

   iv) α^3, β^3 v) $\frac{1}{\alpha^3}, \frac{1}{\beta^3}$ vi) $\alpha + \frac{1}{\alpha}$