8 Mathematical Induction and Binomial Theorem

8.1 Introduction

Francesco Mourolico (1494-1575) devised the method of induction and applied this device first to prove that the sum of the first n odd positive integers equals n^2. He presented many properties of integers and proved some of these properties using the method of *mathematical induction.*

We are aware of the fact that even one exception or case to a mathematical formula is enough to prove it to be false. Such a case or exception which fails the mathematical formula or statement is called a counter example.

The validity of a formula or statement depending on a variable belonging to a certain set is established if it is true for each element of the set under consideration.

For example, we consider the statement $S(n)=n^2-n+41$ is a prime number for every natural number n. The values of the expression n^2-n+41 for some first natural numbers are given in the table as shown below:

n	1	2	3	4	5	6	7	8	9	10	11
$S(n)$	41	43	47	53	61	71	83	97	113	131	151

From the table, it appears that the statement $S(n)$ has enough chance of being true. If we go on trying for the next natural numbers, we find $n=41$ as a counter example which fails the claim of the above statement. So we conclude that to derive a general formula without proof from some special cases is not a wise step. This example was discovered by Euler (1707-1783).

Now we consider another example and try to formulate the result. Our task is to find the sum of the first n odd natural numbers. We write first few sums to see the pattern of sums.

n (The number of terms) Sum

1-- $1=1^2$

2-- $1+3=4=2^2$

3 -- $1+3+5= 9=3^2$

4 -- $1+3+5+7=16=4^2$

5 -- $1+3+5+7+9=25=5^2$

6 -- $1+3+5+7+9+11=36=6^2$

The sequence of sums is $(1)^2,(2)^2,(3)^2,(4)^2,\ldots$

We see that each sum is the square of the number of terms in the sum. So the following statement seems to be true.

For each natural number n,

$$1+3+5+\ldots+(2n-1) = n^2 \quad \ldots \text{(i)} \qquad (\because n\text{th term}=1+(n-1)2)$$

But it is not possible to verify the statement (i) for each positive integer n, because it involves infinitely many calculations which never end.

The method of mathematical induction is used to avoid such situations. Usually it is used to prove the statements or formulae relating to the set $\{1,2,3,\ldots\}$ but in some cases, it is also used to prove the statements relating to the set $\{0,1,2,3,\ldots\}$.

8.2 Principle of Mathematical Induction

The principle of mathematical induction is stated as follows:

If a proposition or statement $S(n)$ for each positive integer n is such that

1) $S(1)$ is true i.e., $S(n)$ is true for $n = 1$ and

$S(k+1)$ is true whenever $S(k)$ is true for any positive integer k, then true for all positive integers.

Procedure:

1. Substituting $n = 1$, show that the statement is true for $n = 1$.
2. Assuming that the statement is true for any positive integer k, then show that it is true for the next higher integer.

For the second condition, one of the following two methods can be used:

M_1: Starting with one side of $S(k+1)$, its other side is derived by using $S(k)$.

M_2: $S(k+1)$ is established by performing algebraic operations on $S(k)$.

Example1: Use mathematical induction to prove that $3+6+9+\ldots+3n=\dfrac{3n(n+1)}{2}$ for every positive integer n.

Solution: Let $S(n)$ be the given statement, that is,

$$S(n): \quad 3+6+9+\ldots+3n = \frac{3n(n+1)}{2} \qquad \text{(i)}$$

1. When $n = 1$, $S(1)$ becomes

$$S(1): 3 = \frac{3(1)(1+1)}{2} = 3$$

Thus $S(1)$ is true i.e., condition (1) is satisfied.

2. Let us assume that $S(n)$ is true for any $n = k \in N$, that is,

$$3+6+9+\ldots+3k = \frac{3k(k+1)}{2} \qquad \text{(A)}$$

The statement for $n=k+1$ becomes

$$3+6+9+\ldots+3k+3(k+1) = \frac{3(k+1)[(k+1)+1]}{2}$$

$$= \frac{3(k+1)(k+2)}{2} \qquad \text{(B)}$$

Adding $3(k+1)$ on both the sides of (A) gives

$$3+6+9+\ldots+3k+3(k+1) = \frac{3k(k+1)}{2} + 3(k+1)$$

$$= 3(k+1)\left(\frac{k}{2}+1\right)$$

$$= \frac{3(k+1)(k+2)}{2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so the condition (2) is satisfi

Since both the conditions are satisfied, therefore, $S(n)$ is true for itive integer n.

8.3 Binomial Theorem

An algebraic expression consisting of two terms such as $a+x$, $x-2y$, $ax+b$ etc., is called a binomial or a binomial expression.

We know by actual multiplication that

$$(a+x)^2 = a^2+2ax+x^2 \qquad \text{(i)}$$

$$(a+x)^3 = a^3+3a^2x+3ax^2+x^3 \qquad \text{(ii)}$$

The right sides of (i) and (ii) are called binomial expansions of the binomial $a+x$ for the indices 2 and 3 respectively.

In general, the rule or formula for expansion of a binomial raised to any positive integral power n is called the binomial theorem for positive integral index n. For any positive integer n,

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \dots + \binom{n}{r-1}a^{n-(r-1)}x^{r-1}$$

$$+\binom{n}{r}a^{n-r}x^r + \dots + \binom{n}{n-1}ax^{n-1} + \binom{n}{n}x^n \qquad \text{(A)}$$

or briefly

$$(a+x)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}x^r$$

where a and x are real numbers.

The rule of expansion given above is called the binomial theorem and it also holds if a or x is complex.

Now we prove the Binomial theorem for any positive integer n, using the principle of mathematical induction.

Proof: Let $S(n)$ be the statement given above as (A).

1. If $n=1$, we obtain

$$S(1):\ (a+x)^1 = \binom{1}{0}a^1 + \binom{1}{1}a^{1-1}x = a+x \qquad \text{which is true}$$

Thus condition (1) is satisfied.

2. Let us assume that the statement is true for any $n = k \in N$, then

$$(a+x)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}x + \binom{k}{2}a^{k-2}x^2 + \dots + \binom{k}{r-1}a^{k-(r-1)}x^{r-1} + \binom{}{r}$$

$$+\dots+\binom{k}{k-1}ax^{k-1}+\binom{k}{k}x^k \qquad \text{(B)}$$

$$S(k+1):(a+k)^{k+1}=\binom{k+1}{0}a^{k+1}+\binom{k+1}{1}a^k\times x+\binom{k+1}{2}a^{k-1}\times x^2+\dots+$$

$$\binom{k+1}{r-1}a^{k-r+2}\times x^{r-1}+\binom{k+1}{r}a^{k-r-1}\times x^r+\dots+\binom{k+1}{k}a\times x^k+\binom{k+1}{k+1}x^{k+1} \qquad \text{(C)}$$

Multiplying both sides of equation (B) by $(a+x)$, we have

$$(a+x)(a+x)^k=(a+x)\Bigg[\binom{k}{0}a^k+\binom{k}{1}a^{k-1}x+\binom{k}{2}a^{k-2}x^2+\dots+\binom{k}{r-1}a^{k-r+1}x^{r-1}$$

$$+\binom{k}{r}a^{k-r}x^r+\dots+\binom{k}{k-1}ax^{k-1}+\binom{k}{k}x^k\Bigg]$$

$$=\Bigg[\binom{k}{0}a^{k+1}+\binom{k}{1}a^kx+\binom{k}{2}a^{k-1}x^2+\dots+\binom{k}{r-1}a^{k-r+2}x^{r-1}$$

$$+\binom{k}{r}a^{k-r+1}x^r+\dots+\binom{k}{k-1}a^2x^{k-1}+\binom{k}{k}ax^k\Bigg]$$

$$+\Bigg[\binom{k}{0}a^kx+\binom{k}{1}a^{k-1}x^2+\binom{k}{2}a^{k-2}\times x^3+\dots+\binom{k}{r-1}a^{k-r-1}\times x^r$$

$$+\binom{k}{r}a^{k-r}x^{r+1}+\dots+\binom{k}{k-1}ax^k+\binom{k}{k}x^{k+1}\Bigg]$$

$$=\binom{k}{0}a^{k+1}+\left[\binom{k}{1}+\binom{k}{0}\right]a^kx+\left[\binom{k}{2}+\binom{k}{1}\right]a^{k-1}x^2+\dots$$

$$+\left[\binom{k}{r}+\binom{k}{r-1}\right]a^{k-r+1}x^r+\dots+\left[\binom{k}{k}+\binom{k}{k-1}\right]ax^k+\binom{k}{k}x^{k+1}$$

As $\binom{k}{0}=\binom{k+1}{0}$, $\binom{k}{k}=\binom{k+1}{k+1}$ and $\binom{k}{r}+\binom{k}{r-1}=\binom{k+1}{r}$ for $1\le r\le k$

$$\therefore (a+x)^{k+1}=\binom{k+1}{0}a^{k+1}+\binom{k+1}{1}a^kx+\binom{k+1}{2}a^{k-1}x^2+\dots$$

$$+\binom{k+1}{r}a^{k-r+1}x^r+\dots+\binom{k+1}{k}a\,x^k+\binom{k+1}{k+1}x^{k+1}$$

We find that if the statement is true of $n=k$, then it is also true for $n=k+$

... that the statement is true for all positive integral values of n

Thus $S(2)$ is true, i.e., the first condition is satisfied.

2. Let the statement be true for any $n = k(\geq 2) \in Z$, that is

$4^k > 3^k + 4$ (A)

Multiplying both sides of inequality (A) by 4, we get

or $4 \cdot 4^k > 4(3^k + 4)$

or $4^{k+1} > (3+1)3^k + 16$

or $4^{k+1} > 3^{k+1} + 4 + 3^k + 12$

or $4^{k+1} > 3^{k+1} + 4$ $(\because 3^k + 12 > 0)$ (B)

The inequality (B), satisfies the condition (2).

Since both the conditions are satisfied, therefore, by the principle of extended mathematical induction, the given inequality is true for all integers $n \geq 2$.

Exercise 8.1

Use mathematical induction to prove the following formulae for every positive integer n.

1. $1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1)$
2. $1 + 3 + 5 + \ldots + (2n - 1) = n^2$
3. $1 + 4 + 7 + \ldots + (3n - 2) = \dfrac{n(3n - 1)}{2}$
4. $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$
5. $1 + \dfrac{1}{2} + \dfrac{1}{4} + \ldots + \dfrac{1}{2^{n-1}} = 2\left[1 - \dfrac{1}{2^n}\right]$
6. $2 + 4 + 6 + \ldots + 2n = n(n + 1)$
7. $2 + 6 + 18 + \ldots + 2 \times 3^{n-1} = 3^n - 1$
8. $1\times3 + 2\times5 + 3\times7 + \ldots + n\times(2n+1) = \dfrac{n(n+1)(4n+5)}{6}$
9. $1\times2 + 2\times3 + 3\times4 + \ldots + n\times(n+1) = \dfrac{n(n+1)(n+2)}{3}$
10. $1\times2 + 3\times4 + 5\times6 + \ldots + (2n-1)\times2n = \dfrac{n(n+1)(4n-1)}{3}$

9 Fundamentals of Trigonometry

9.1 Introduction

Trigonometry is an important branch of Mathematics. The word **Trigonometry** has been derived from three Greek words: ***Trei*** (three), ***Goni*** (angles) and **Metron** (measurement). Literally it means **measurement of triangle.**

For study of calculus it is essential to have a sound knowledge of trigonometry.

It is extensively used in Business, Engineering, Surveying, Navigation, Astronomy, Physical and Social Sciences.

9.2 Units of Measures of Angles

Concept of an Angle

Two rays with a common starting point form an angle. One of the rays of angle is called initial side and the other as terminal side. The angle is identified by showing the direction of rotation from the initial side to the terminal side.

An angle is said to be positive/negative if the rotation is anti-clockwise/ clockwise. Angles are usually denoted by Greek letters such as α (alpha), β (beta), γ (gamma), θ (theta) etc.

In figure 9.1 $\angle AOB$ is positive and $\angle COD$ is negative.

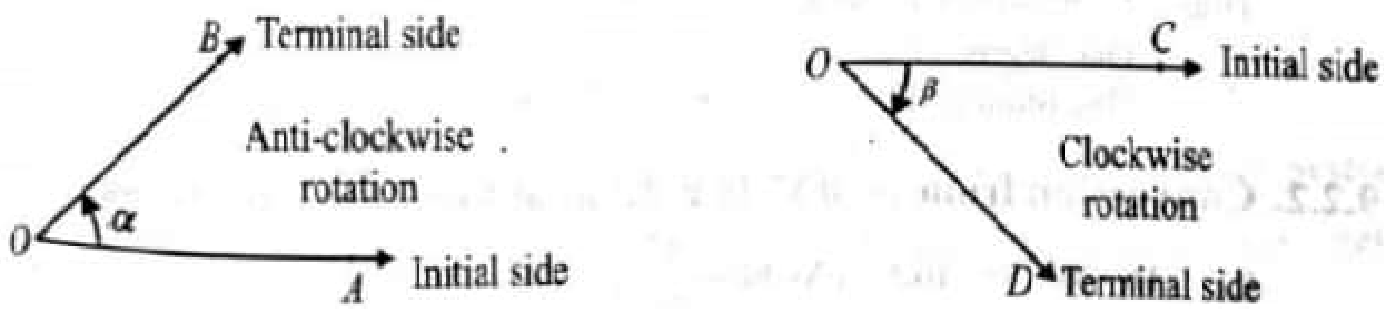


figure 9.1

∴ There are two commonly used measurements for angles: **Degrees and Radians.**

which are explained as below:

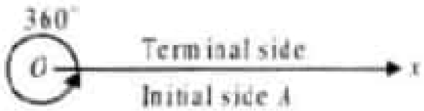
9.2.1. Sexagesimal System: (Degree, Minute and Second).

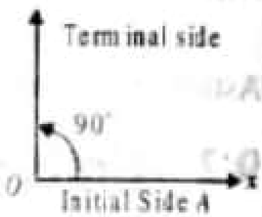
If the initial ray $\overrightarrow{OA}$ rotates in anti-clockwise direction in such a way that it coincides with itself, the angle then formed is said to be of 360 degrees (360°).

One rotation (anti-clockwise) = 360°

$\frac{1}{2}$ rotation (anti-clockwise) = 180° is called a straight angle

$\frac{1}{4}$ rotation (anti-clockwise) = 90° is called a right angle.





1 rotation $=360^\circ$ $\qquad$ $\frac{1}{2}$ rotation $= 180^\circ$ $\qquad$ $\frac{1}{4}$ rotation $= 90^\circ$

1 degree (1°) is divided into 60 minutes ($60'$) and 1 minute ($1'$) is divided into 60 seconds ($60''$). As this system of measurement of angle owes its origin to the English and because 90, 60 are multiples of 6 and 10, so it is known as English system or Sexagesimal system

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Thus

1 rotation (anti-clockwise)	=	360°.
One degree (1°)	=	$60'$
One minute ($1'$)	=	$60''$

9.2.2. Conversion from $D^\circ M' S''$ to a decimal form and vice versa.

(i) $16^\circ\,30' = 16.5^\circ$ $\left(\text{As } 30' = \frac{1}{2}{}^\circ = 0.5^\circ\right)$

(ii) $45.25^\circ = 45^\circ\,15'$ $\left(0.25^\circ = \frac{25^\circ}{100} = \frac{1^\circ}{4} = \frac{60'}{4} = 15'\right)$

Example 1: Convert $18^\circ\ 6'\ 21''$ to decimal form.

Solution: $1' = \left(\dfrac{1}{60}\right)^\circ$ and $1'' = \left(\dfrac{1}{60}\right)' = \left(\dfrac{1}{60\times 60}\right)^\circ$

$$\therefore 18^\circ\ 6'\ 21'' = \left[18 + 6\left(\frac{1}{60}\right) + 21\left(\frac{1}{60\times 60}\right)\right]^\circ$$

$$= (18 + 0.1 + 0.005833)^\circ = 18.105833^\circ$$

Example 2 : Convert 21.256° to the $D^\circ M'S''$ form

Solution: $0.256^\circ = (0.256)(1^\circ)$

$= (0.256)(60') = 15.36'$

and $0.36' = (0.36)(1')$

$= (0.36)(60'') = 21.6''$

Therefore,

$21.256^\circ = 21^\circ + 0.256^\circ$

$= 21^\circ + 15.36'$

$= 21^\circ + 15' + 0.36'$

$= 21^\circ + 15' + 21.6''$

$= 21^\circ\ 15'\ 22''$ rounded off to nearest second

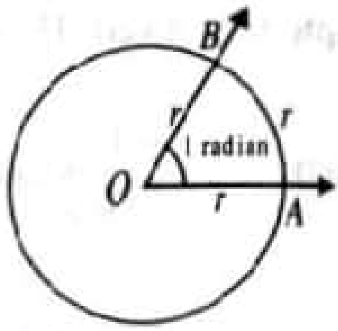
9.2.3. Circular System (Radians)

There is another system of angular measurement, called the **Circular System.** It is most useful for the study of higher mathematics. Specially in Calculus, angles are measured in radians.

Definition: Radian is the measure of the angle subtended at the center of the circle by an arc, whose length is equal to the radius of the circle.

Consider a circle of radius r. Construct an angle $\angle AOB$ at the centre of the circle whose rays cut off an arc $\widehat{AB}$ on the circle whose length is equal to the radius r.

Thus $m\angle AOB = 1$ radian.

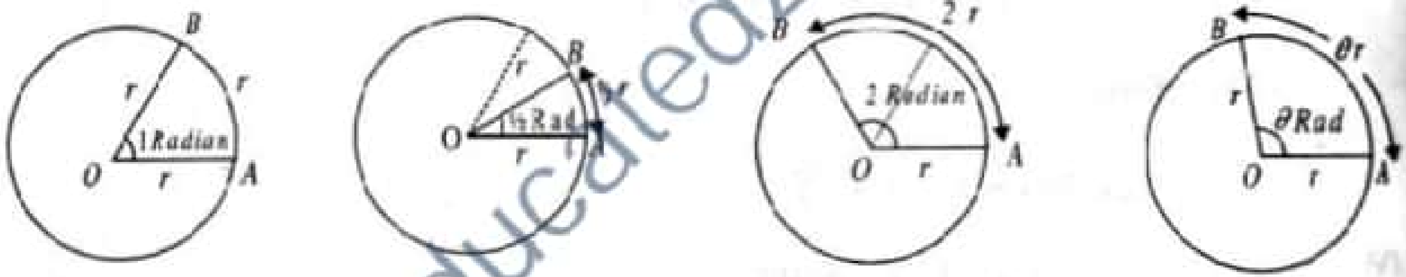


9.3 Relation between the length of an arc of a circle and the circular measure of its central angle.

$$\text{Prove that } \theta = \frac{l}{r},$$

where r is the radius of the circle l, is the length of the arc and θ is the circular measure of the central angle.

Proof:



By definition of radian;

An angle of 1 radian subtends an arc $\widehat{AB}$ on the circle of length $= 1.r$

An angle of $\frac{1}{2}$ radian subtends an arc $\widehat{AB}$ on the circle of length $= \frac{1}{2}.r$

An angle of 2 radians subtends an arc $\widehat{AB}$ on the circle of length $= 2.r$

$\therefore$ An angle of θ radian subtends an arc $\widehat{AB}$ on the circle of length $= \theta.r$

$$\Rightarrow \quad \widehat{AB} = \theta.r$$

$$\Rightarrow \quad l = \theta.r$$

$$\therefore \quad \theta = \frac{l}{r}$$

9.8 Fundamental Identities

For any real number θ, we shall derive the following three fundamental identities:

i) $\sin^2\theta + \cos^2\theta \;=\; 1$

ii) $1 + \tan^2\theta \;=\; \sec^2\theta$

iii) $1 + \cot^2\theta \;=\; \csc^2\theta.$

Proof:

(i) Refer to right triangle ABC in fig. (1) by Pythagoras theorem, we have

Dividing $a^2+b^2=c^2$ both sides by c^2, we get

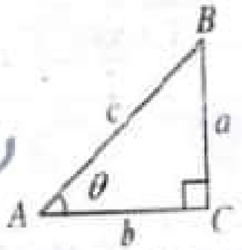


Fig.(1)

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$

$$\Rightarrow \quad \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$\Rightarrow \quad (\sin\theta)^2 + (\cos\theta)^2 = 1$$

$$\therefore \quad \boxed{\sin^2\theta + \cos^2\theta = 1} \qquad (1)$$

ii) Again as $a^2 + b^2 = c^2$

Dividing both sides by b^2, we get

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$$

$$\Rightarrow \quad \left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$$

$$\Rightarrow \quad (\tan\theta)^2 + 1 = (\sec\theta)^2 \qquad (2)$$

$$\Rightarrow \quad \boxed{1 + \tan^2\theta = \sec^2\theta}$$

ii) Again as $a^2 + b^2 = c^2$

Dividing both sides by b^2, we get

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$$

$$\Rightarrow \left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$$

$$\Rightarrow (\tan\theta)^2 + 1 = (\sec\theta)^2$$

$$\Rightarrow \boxed{1 + \tan^2\theta = \sec^2\theta} \qquad (2)$$

(iii) Again as $a^2 + b^2 = c^2$

Dividing both side by a^2, we get

$$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}$$

$$\Rightarrow 1 + \left(\frac{b}{a}\right)^2 = \left(\frac{c}{a}\right)^2$$

$$\Rightarrow 1 + (\cot\theta)^2 = (\csc\theta)^2$$

$$\therefore \boxed{1 + \cot^2\theta = \csc^2\theta} \qquad (3)$$

Note: $(\sin\theta)^2 = \sin^2\theta$, $(\cos\theta)^2 = \cos^2\theta$ and $(\tan\theta)^2 = \tan^2\theta$ etc.