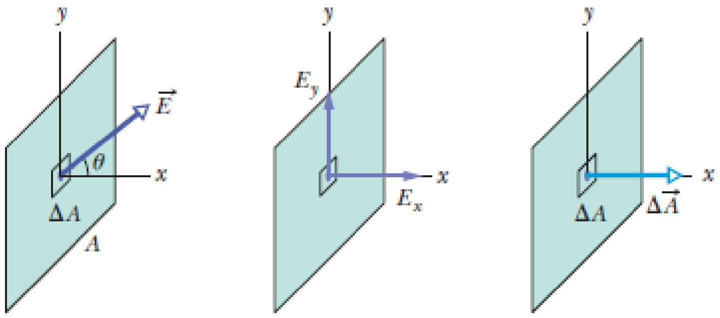
Electric Flux

Flat Surface, Uniform Field. We begin with a flat surface with area A in a uniform electric field . Figure *a* shows one of the electric field vectors $\bar{E}$ piercing a small square patch with area ΔA (where Δ indicates "small"). Actually, only the x component (with magnitude $E_x = E\cos\theta$ in Fig. *b*) pierces the patch. The y component merely skims along the surface (no piercing in that) and does not come into play in Gauss' law. The *amount* of electric field piercing the patch is defined to be the **electric flux** $\Delta\phi$ through it:

$$\Delta\Phi = (E\cos\theta)\,\Delta A.$$



(a) (b) (c)

There is another way to write the right side of this statement so that we have only the piercing component of $\bar{E}$. We define an area vector $\Delta\bar{A}$ that is perpendicular to the patch and that has a magnitude equal to the area ΔA of the patch (fig. *c*).
Then we can write

$$\Delta\Phi = \vec{E}\cdot\Delta\vec{A},$$

and the dot product automatically gives us the component $\bar{E}$ of that is parallel to $\Delta\bar{A}$ and thus piercing the patch.

To find the total flux through the surface in Figure, we sum the flux through every patch on the surface:

$$\Phi = \sum \vec{E}\cdot\Delta\vec{A}.$$

However, because we do not want to sum hundreds (or more) flux values, we transform the summation into an integral by shrinking the patches from small squares with area $\Delta\bar{A}$ to *patch elements* (or *area elements*) with area dA. The total flux is then

$$\Phi = \int \vec{E}\cdot d\vec{A} \qquad \text{(total flux).}$$

Directions: To keep track of the piercing direction, we again use an area vector $\Delta\bar{A}$ that is perpendicular to a patch, but now we always draw it pointing outward from the surface (*away from the interior*). Then if a field vector pierces outward, it and the area vector are in the same direction, the angle is $\theta = 0$, and $\cos\theta = 1$.

Thus, the dot product $\bar{E}.\Delta\bar{A}$ is positive and so is the flux. Conversely, if a field vector pierces inward, the angle is $\theta = 180°$ and $\cos\theta = -1$. Thus, the dot product is negative and so is the flux. If a field vector skims the surface (no piercing), the dot product is zero (because $\cos 90° = 0$) and so is the flux.

An inward piercing field is negative flux. An outward piercing field is positive flux. A skimming field is zero flux.

***Unit*:** Note that flux is a scalar (yes, we talk about field vectors but flux is the *amount* of piercing field, not a vector itself). The SI unit of flux is the newton–square-meter per coulomb ($N.m^2/C$).

Gauss' Law

Gauss' law relates the net flux ϕ of an electric field through a closed surface (a Gaussian surface) to the *net* charge q_{enc} that is *enclosed* by that surface. It tells us that

$$\varepsilon_0 \Phi = q_{enc} \quad \text{(Gauss' law).}$$

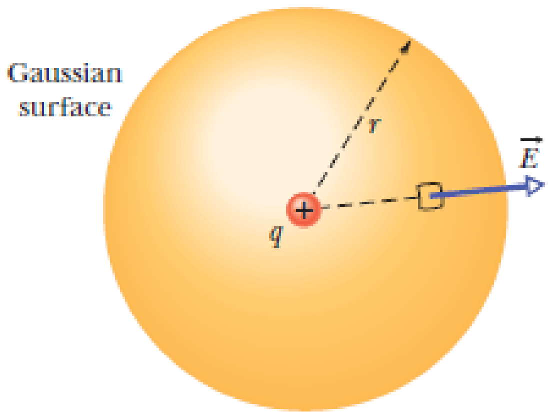
By substituting $\phi = \oint E.dA$, the definition of flux, we can also write Gauss' law as

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc} \quad \text{(Gauss' law).}$$

Above equations hold only when net charge is located in a vacuum or (what is the same for most practical purposes) in air.

The net charge q_{enc} is the algebraic sum of all the *enclosed* positive and negative charges, and it can be positive, negative, or zero. We include the sign, rather than just use the magnitude of the enclosed charge, because the sign tells us something about the net flux through the Gaussian surface: If q_{enc} is positive, the net flux is *outward*; if q_{enc} is negative, the net flux is *inward.*

We enclose the particle in a Gaussian sphere that is centered on the particle, as shown in Fig. for a particle with positive charge q. Then the electric field has the same magnitude E at any point on the sphere (all points are at the same distance r). That feature will simplify the integration. The drill here is the same as previously.



Pick a patch element on the surface and draw its area vector perpendicular to the patch and directed outward. From the symmetry of the situation, we know that the electric field E at the patch is also radially outward and thus at an angle $\theta = 0$ with $d\bar{A}$. So we rewrite Gauss' law as

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 \oint E\, dA = q_{\text{enc}}.$$

Here $q_{\text{enc}} = q$. Because the field magnitude E is the same at every patch element, E can be pulled outside the integral

$$\varepsilon_0 E \oint dA = q.$$

The remaining integral is just an instruction to sum all the areas of the patch elements on the sphere, but we already know that the total area is $4\pi r^2$. Substituting this, we have

$$\varepsilon_0 E(4\pi r^2) = q$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}.$$

This is exactly which we found using Coulomb's law.