Ch.9: Normal Distribution

Let "X" be a continuous random variable with interval $(-\infty,+\infty)$ is said to be normal distribution having its probability density function (p.d.f) is given as

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \qquad -\infty\le X\le+\infty$$

It has two parameters (μ,σ^2).

Where $\mu = Mean$ $\quad \sigma^2 = Variance$ $\quad \sigma = S\tan dard\ deviation$

$\pi = Cons\tan t\ Approximately\ equal\ to = \frac{22}{7} = 3.14159$

$e = Cons\tan t\ Approximately\ equal\ to = 2.71828$

$X = Abscissa\ \ i.e.\ value\ marked\ on\ X-axis$

$Y = Ordinate\ hieght\ \ i.e.\ value\ marked\ on\ Y-axis$

Properties of normal distribution

i) It is continuous distribution and its range $(-\infty,+\infty)$
ii) Total area under the curve is unity
iii) It is bell shape distribution
iv) It is symmetrical distribution and its mean, median and mode are identical
v) It is unimodal distribution and maximum ordinate of the curve at $X=\mu$ is $\frac{1}{\sigma\sqrt{2\pi}}$
vi) It has two points of inflection which are equidistant from mean" μ "are $\left(\mu-\sigma,\frac{1}{\sigma\sqrt{2\pi e}}\right)$ and $\left(\mu+\sigma,\frac{1}{\sigma\sqrt{2\pi e}}\right)$
vii) The mean deviation of normal distribution is approximately $\frac{4}{5}$ of its Standard deviation i.e. $M.D=\frac{4}{5}\sigma=0.7979\sigma$
viii) The Quartile deviation of normal distribution is approximately $\frac{2}{3}$ of its Standard deviation i.e. $Q.D=\frac{2}{3}\sigma=0.6745\sigma$
ix) All odd order moments about mean equal to zero i.e. $\mu_1=\mu_3=\mu_5=....=0$
x) The expression of even order moments about mean are $\mu_{2n}=\left(\frac{\sigma^2}{2}\right)^n\frac{(2n)!}{n!}$
xi) If X is $N(\mu,\sigma^2)$ and if $Y=a+bX$ then Y is $N(a+\mu b,b^2\sigma^2)$
xii) The sum of independent normal variables is a normal variable
xiii) The property of normal distribution
i) $P(\mu-0.6745\sigma<X<\mu+0.6745\sigma)=0.50\ or\ 50\%$
ii) $P(\mu-\sigma<X<\mu+\sigma)=0.6827\ or\ 68.27\%$
iii) $P(\mu-2\sigma<X<\mu+2\sigma)=0.9544\ or\ 95.44\%$
iv) $P(\mu-3\sigma<X<\mu+3\sigma)=0.9973\ or\ 95.73\%$
xiv) The two quartiles are equidistant from the mean
$Q_1=\mu-0.6745\sigma$ Lower quartile
$Q_3=\mu+0.6745\sigma$ Upper quartile
xv) The normal curve approaches but never touches the base line. So the curve is asymptotic to the horizontal line.
xvi) It has two parameters (μ,σ^2)

Standard normal variate

Ans: Any variable having zero mean and unit variance is called standard normal variate or variable. i.e. $Z=\frac{X-\mu}{\sigma}$

Standard normal distribution
The normal probability distribution of "Z" which has zero mean and unit variance is called the standardized normal distribution. It is denoted by $Z \to N(0,1)$ and its p.d.f

given as $f(Z) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}Z^2}$ $\qquad -\infty \le Z \le +\infty$

Standard normal distribution

Properties of standard normal distribution

i) It is continuous distribution and its range $(-\infty, +\infty)$
ii) Total area under the curve is unity
iii) It is bell shape distribution
iv) It is symmetrical distribution and its mean, median and mode ere equal to zero
v) The mean deviation of standard normal distribution is approximately $\frac{4}{5}$ i.e.

$$M.D = \frac{4}{5} = 0.7979$$

vi) The Quartile deviation of standard normal distribution is approximately $\frac{2}{3}$

i.e. $Q.D = \frac{2}{3} = 0.6745$
vii) All odd order moments about mean equal to zero i.e. $\mu_1 = \mu_3 = \mu_5 = ... = 0$
viii) The two quartiles of standard normal distribution
$Q_1 = -0.6745$ Lower quartile
$Q_3 = 0.6745$ Upper quartile
ix) It has two parameters $Z \to N(0,1)$

Theorem.No.1: Show that total area under the curve is unity
Proof: Let by definition

$$Total\ \ Area = \int_{-\infty}^{\infty} f(x)d(x)$$

$$X \to N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \qquad -\infty \le X \le \infty$$

$$Total\ \ Area = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} d(x)$$

$$Total\ \ Area = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} d(x) \qquad \text{(i)}$$

Put $Z = \dfrac{X-\mu}{\sigma}$

$Z\sigma = x - \mu$
$X = Z\sigma + \mu$
$d(x) = Zd(z)$ And Limits remains same $-\infty \le Z \le \infty$

$$Total\ \ Area = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}Z^2} \sigma\ d(z)$$

$$Total\ \ Area = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}Z^2} d(z) \qquad \text{(ii)}$$

As $f(z) = e^{-\frac{1}{2}Z^2}$
Put $Z = -Z$

$f(-Z) = e^{-\frac{1}{2}(-Z)^2} = e^{-\frac{1}{2}Z^2} = f(Z)$ Its even function then eq.(ii) becomes

$$Total\ \ Area = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}Z^2} d(z) \qquad \text{(iii)}$$

Put $t = \frac{1}{2}Z^2$

$\sqrt{2t} = Z$

$\frac{1}{2}(2t)^{-\frac{1}{2}} 2d(t) = d(z)$

$(2t)^{-\frac{1}{2}} d(t) = d(z)$ And Limits remains same $0 \le t \le \infty$

$$Total\ \ Area = \frac{2}{\sqrt{2\pi}} \int_0^\infty (2t)^{-\frac{1}{2}} e^{-t}\ d(t)$$

$$Total\ \ Area = \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{t^{\frac{1}{2}-1}}{2^{\frac{1}{2}}} e^{-t}\ d(t)$$

$$Total\ \ Area = \frac{2}{\sqrt{2}\sqrt{\pi}} \int_0^\infty \frac{t^{\frac{1}{2}-1}}{\sqrt{2}} e^{-t}\ d(t)$$

$$Total\ \ Area = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{\frac{1}{2}-1} e^{-t}\ d(t) \qquad \text{(iv)}$$

As we know that Gamma function is

$$\overline{)\alpha}\beta^{\alpha} = \int_0^\infty t^{\alpha-1} e^{-t/\beta}\ d(t) \qquad \text{v)}$$

Comparing (iv) and (v) then we get

$\alpha = \frac{1}{2}$ $\quad And \quad \beta = 1$ Then eq.(iv) becomes

$Total\ \ Area = \frac{1}{\sqrt{\pi}} \overline{)\frac{1}{2}} 1^{\frac{1}{2}} = \frac{1}{\sqrt{\pi}} \overline{)\frac{1}{2}} = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1$ Therefore $\overline{)\frac{1}{2}} = \sqrt{\pi}$

$Total\ \ Area = 1$ Hence proved

Theorem.No.2: Show that mean of normal distribution is $"\mu"$

Proof: Let by definition

$$E(X) = \int_{-\infty}^{\infty} Xf(x)d(x)$$

$X \to N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \qquad -\infty \le X \le \infty$$

$$E(X) = \int_{-\infty}^{\infty} X \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} d(x)$$

$$E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} Xe^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} d(x) \qquad \text{(i)}$$

Put $\quad Z = \frac{X-\mu}{\sigma}$

$Z\sigma = x - \mu$

$X = Z\sigma + \mu$

$d(x) = \sigma\ d(z)$ And Limits remains same $-\infty \le Z \le \infty$

$$E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma Z + \mu) e^{-\frac{1}{2}Z^2}\ \sigma\ d(z)$$

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma Z e^{-\frac{1}{2}Z^2}\ d(z) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu e^{-\frac{1}{2}Z^2}\ d(z)$$

$$E(X) = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z e^{-\frac{1}{2}Z^2}\ d(z) + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}Z^2}\ d(z) \qquad \text{(ii)}$$

As $f(z)=e^{-\frac{1}{2}Z^2}$ $\quad f(z)=Ze^{-\frac{1}{2}Z^2}$

Put $Z=-Z$

$$f(-Z)=(-Z)e^{-\frac{1}{2}(-Z)^2}=-Ze^{-\frac{1}{2}Z^2}=-f(Z)$$

$$f(-Z)=e^{-\frac{1}{2}(-Z)^2}=e^{-\frac{1}{2}Z^2}=f(Z)$$

1st function is odd and 2nd is even function then eq. (ii) becomes

$$E(X)=\frac{\sigma}{\sqrt{2\pi}}(0)+\frac{2\mu}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}Z^2}\ d(z)$$

$$E(X)=\frac{2\mu}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}Z^2}\ d(z) \qquad \text{(iii)}$$

Put $t=\frac{1}{2}Z^2$

$\sqrt{2t}=Z$

$\frac{1}{2}(2t)^{-\frac{1}{2}}2d(t)=d(z)$

$(2t)^{-\frac{1}{2}}d(t)=d(z)$ And Limits remains same $0\le t\le\infty$

$$E(X)=\frac{2\mu}{\sqrt{2\pi}}\int_0^\infty (2t)^{-\frac{1}{2}}e^{-t}\ d(t)$$

$$E(X)=\frac{2\mu}{\sqrt{2\pi}}\int_0^\infty \frac{t^{\frac{1}{2}-1}}{2^{\frac{1}{2}}}e^{-t}\ d(t)$$

$$E(X)=\frac{2\mu}{\sqrt{2}\sqrt{\pi}}\int_0^\infty \frac{t^{\frac{1}{2}-1}}{\sqrt{2}}e^{-t}\ d(t)$$

$$E(X)=\frac{\mu}{\sqrt{\pi}}\int_0^\infty t^{\frac{1}{2}-1}e^{-t}\ d(t) \qquad \text{(iv)}$$

As we know that Gamma function is

$$\Gamma\alpha\,\beta^\alpha=\int_0^\infty t^{\alpha-1}e^{-t/\beta}\ d(t) \qquad \text{v)}$$

Comparing (iv) and (v) then we get

$\alpha=\frac{1}{2}$ *And* $\beta=1$ Then eq. (iv) becomes

$$E(X)=\frac{\mu}{\sqrt{\pi}}\Gamma\frac{1}{2}1^{\frac{1}{2}}=\frac{\mu}{\sqrt{\pi}}\Gamma\frac{1}{2}=\frac{\mu}{\sqrt{\pi}}\sqrt{\pi}=\mu$$ Therefore $\Gamma\frac{1}{2}=\sqrt{\pi}$

$E(X)=\mu$ Hence proved

Theorem.No.3: Show that Variance of normal distribution is "σ^2"

Proof: Let by definition

$$Var(X)=E[X-\mu]^2=\int_{-\infty}^{\infty}[X-\mu]^2 f(x)d(x)$$

$X\to N(\mu,\sigma^2)$

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \qquad -\infty\le X\le\infty$$

$$Var(X)=\int_{-\infty}^{\infty}[X-\mu]^2\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}d(x)$$

$$Var(X)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}[X-\mu]^2e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}d(x) \qquad \text{(i)}$$

Put $\quad Z = \dfrac{X-\mu}{\sigma}$

$Z\sigma = x - \mu$

$X = Z\sigma + \mu$

$d(x) = \sigma\ d(z)$ And Limits remains same $-\infty \le Z \le \infty$

$$Var(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma Z)^2 e^{-\frac{1}{2}Z^2}\ \sigma\ d(z)$$

$$Var(X) = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z^2 e^{-\frac{1}{2}Z^2}\ d(z) \qquad \text{(ii)}$$

As $f(z) = Z^2 e^{-\frac{1}{2}Z^2}$

Put $Z = -Z$

$$f(-Z) = (-Z)^2 e^{-\frac{1}{2}(-Z)^2} = Z^2 e^{-\frac{1}{2}Z^2} = f(Z)$$

$$f(-Z) = e^{-\frac{1}{2}(-Z)^2} = e^{-\frac{1}{2}Z^2} = f(Z)$$

It is even function then eq. (ii) becomes

$$Var(X) = \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} Z^2 e^{-\frac{1}{2}Z^2}\ d(z) \qquad \text{(iii)}$$

Put $t = \dfrac{1}{2}Z^2$

$2t = Z^2$

$\sqrt{2t} = Z$

$\dfrac{1}{2}(2t)^{-\frac{1}{2}} 2d(t) = d(z)$

$(2t)^{-\frac{1}{2}} d(t) = d(z)$ And Limits remains same $0 \le t \le \infty$

$$Var(X) = \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} 2t(2t)^{-\frac{1}{2}} e^{-t}\ d(t)$$

$$Var(X) = \frac{4\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{t^{\frac{1}{2}}}{2^{\frac{1}{2}}} e^{-t}\ d(t)$$

$$Var(X) = \frac{4\sigma^2}{\sqrt{2}\sqrt{\pi}} \int_{0}^{\infty} \frac{t^{\frac{1}{2}-1}}{\sqrt{2}} e^{-t}\ d(t)$$

$$Var(X) = \frac{4\sigma^2}{2\sqrt{\pi}} \int_{0}^{\infty} t^{\frac{1}{2}-1} e^{-t}\ d(t)$$

$$Var(X) = \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} t^{\frac{3}{2}-1} e^{-t}\ d(t) \qquad \text{(iv)}$$

As we know that Gamma function is

$$\Gamma\alpha\, \beta^{\alpha} = \int_{0}^{\infty} t^{\alpha-1} e^{-t/\beta}\ d(t) \qquad \text{v)}$$

Comparing (iv) and (v) then we get

$\alpha = \dfrac{3}{2}$ $\quad And \quad \beta = 1$ Then eq. (iv) becomes

$$Var(X) = \frac{\sigma^2}{\sqrt{\pi}} \Gamma\frac{3}{2}\, 1^{\frac{3}{2}} = \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}+1\right)$$

$$Var(X) = \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \Gamma\frac{1}{2} = \frac{\sigma^2}{\sqrt{\pi}} \sqrt{\pi} = \sigma^2 \qquad \text{Therefore} \quad \Gamma\frac{1}{2} = \sqrt{\pi}$$

$Var(X) = \sigma^2$ Hence proved

Theorem.No.4: Show that Mean deviation of normal distribution is "$M.D=\frac{4}{5}\sigma$"

Proof: Let by definition

$$M.D=E|X-\mu|=\int_{-\infty}^{\infty}|X-\mu|f(x)d(x)$$

$$X\rightarrow N(\mu,\sigma^2)$$

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \qquad -\infty\le X\le\infty$$

$$M.D=\int_{-\infty}^{\infty}|X-\mu|\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}d(x)$$

$$M.D=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}|X-\mu|e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}d(x) \qquad \text{(i)}$$

Put $Z=\frac{X-\mu}{\sigma}$

$$Z\sigma=x-\mu$$

$$X=Z\sigma+\mu$$

$d(x)=\sigma\ d(z)$ And Limits remains same $-\infty\le Z\le\infty$

$$M.D=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}|\sigma Z|e^{-\frac{1}{2}Z^2}\ \sigma\ d(z)$$

$$Var(X)=\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|Z|e^{-\frac{1}{2}Z^2}\ d(z)$$

$$M.D=\frac{\sigma}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}-Ze^{-\frac{1}{2}Z^2}\ d(z)+\int_{0}^{\infty}Ze^{-\frac{1}{2}Z^2}\ d(z)\right]$$

$$M.D=\frac{\sigma}{\sqrt{2\pi}}\left[\int_{0}^{\infty}Ze^{-\frac{1}{2}Z^2}\ d(z)+\int_{0}^{\infty}Ze^{-\frac{1}{2}Z^2}\ d(z)\right]$$

$$M.D=\frac{\sigma}{\sqrt{2\pi}}\left[2\int_{0}^{\infty}Ze^{-\frac{1}{2}Z^2}\ d(z)\right]$$

$$M.D=\frac{2\sigma}{\sqrt{2\pi}}\int_{0}^{\infty}Ze^{-\frac{1}{2}Z^2}\ d(z) \qquad \text{(ii)}$$

Put $t=\frac{1}{2}Z^2$

$$2t=Z^2$$

$$\sqrt{2t}=Z$$

$$\frac{1}{2}(2t)^{-\frac{1}{2}}2d(t)=d(z)$$

$(2t)^{-\frac{1}{2}}d(t)=d(z)$ And Limits remains same $0\le t\le\infty$

$$M.D=\frac{2\sigma}{\sqrt{2\pi}}\int_{0}^{\infty}\sqrt{2t}(2t)^{-\frac{1}{2}}e^{-t}\ d(t)$$

$$M.D=\frac{2\sigma}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{\sqrt{2}\ \ t^{\frac{1}{2}}t^{-\frac{1}{2}}}{2^{\frac{1}{2}}}e^{-t}\ d(t)$$

$$M.D=\frac{2\sigma}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{\sqrt{2}\ \ t^{-\frac{1}{2}+\frac{1}{2}}}{\sqrt{2}}e^{-t}\ d(t)$$

$$M.D=\frac{2\sigma}{\sqrt{2\pi}}\int_{0}^{\infty}t^{0}e^{-t}\ d(t)$$

$$Var(X)=\frac{2\sigma}{\sqrt{2\pi}}\int_0^{\infty}t^{1-1}e^{-t}\ d(t) \qquad \text{(iv)}$$

As we know that Gamma function is

$$\overline{)\alpha}\beta^{\alpha}=\int_0^{\infty}t^{\alpha-1}e^{-t/\beta}\ d(t) \qquad \text{(v)}$$

Comparing (iv) and (v) then we get

$\alpha=1$ *And* $\beta=1$ Then eq. (iv) becomes

$$M.D=\frac{2\sigma}{\sqrt{2\pi}}\overline{)1}=\frac{\sqrt{2}\sqrt{2}\sigma}{\sqrt{2}\sqrt{\pi}}=\frac{\sqrt{2}\sigma}{\sqrt{\pi}}=\sqrt{\frac{2}{\pi}}\ \sigma=0.7979\sigma=\frac{4}{5}\sigma$$ Hence proved

Theorem.No.5: Show that mean, median and mode of normal distribution is equal to μ.

Proof: Let by definition

$$E(X)=\int_{-\infty}^{\infty}Xf(x)d(x)$$

$$X\to N(\mu,\sigma^2)$$

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \qquad -\infty\le X\le\infty$$

$$E(X)=\int_{-\infty}^{\infty}X\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}d(x)$$

$$E(X)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}Xe^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}d(x) \qquad \text{(i)}$$

Put $Z=\frac{X-\mu}{\sigma}$

$Z\sigma=x-\mu$

$X=Z\sigma+\mu$

$d(x)=\sigma\ d(z)$ And Limits remains same $-\infty\le Z\le\infty$

$$E(X)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(\sigma Z+\mu)e^{-\frac{1}{2}Z^2}\ \sigma\ d(z)$$

$$E(X)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sigma Ze^{-\frac{1}{2}Z^2}\ d(z)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mu e^{-\frac{1}{2}Z^2}\ d(z)$$

$$E(X)=\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty}Ze^{-\frac{1}{2}Z^2}\ d(z)+\frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}Z^2}\ d(z) \qquad \text{(ii)}$$

As $f(z)=e^{-\frac{1}{2}Z^2}$ $\qquad f(z)=Ze^{-\frac{1}{2}Z^2}$

Put $Z=-Z$

$$f(-Z)=(-Z)e^{-\frac{1}{2}(-Z)^2}=-Ze^{-\frac{1}{2}Z^2}=-f(Z)$$

$$f(-Z)=e^{-\frac{1}{2}(-Z)^2}=e^{-\frac{1}{2}Z^2}=f(Z)$$

1st function is odd and 2nd is even function then eq. (ii) becomes

$$E(X)=\frac{\sigma}{\sqrt{2\pi}}(0)+\frac{2\mu}{\sqrt{2\pi}}\int_0^{\infty}e^{-\frac{1}{2}Z^2}\ d(z)$$

$$E(X)=\frac{2\mu}{\sqrt{2\pi}}\int_0^{\infty}e^{-\frac{1}{2}Z^2}\ d(z) \qquad \text{(iii)}$$

Put $t=\frac{1}{2}Z^2$

$\sqrt{2t}=Z$

$\frac{1}{2}(2t)^{-\frac{1}{2}}2d(t)=d(z)$

$(2t)^{-\frac{1}{2}}d(t)=d(z)$ And Limits remains same $0\le t\le\infty$

$$E(X)=\frac{2\mu}{\sqrt{2\pi}}\int_0^\infty (2t)^{-\frac{1}{2}} e^{-t}\ d(t)$$

$$E(X)=\frac{2\mu}{\sqrt{2\pi}}\int_0^\infty \frac{t^{\frac{1}{2}-1}}{2^{\frac{1}{2}}} e^{-t}\ d(t)$$

$$E(X)=\frac{2\mu}{\sqrt{2}\sqrt{\pi}}\int_0^\infty \frac{t^{\frac{1}{2}-1}}{\sqrt{2}} e^{-t}\ d(t)$$

$$E(X)=\frac{\mu}{\sqrt{\pi}}\int_0^\infty t^{\frac{1}{2}-1} e^{-t}\ d(t) \qquad \text{(iv)}$$

As we know that Gamma function is

$$\Gamma\alpha\,\beta^{\alpha}=\int_0^\infty t^{\alpha-1}e^{-t/\beta}\ d(t) \qquad \text{v)}$$

Comparing (iv) and (v) then we get

$\alpha=\frac{1}{2}$ $\quad And \quad \beta=1$ Then eq. (iv) becomes

$$E(X)=\frac{\mu}{\sqrt{\pi}}\Gamma\frac{1}{2}1^{\frac{1}{2}}=\frac{\mu}{\sqrt{\pi}}\Gamma\frac{1}{2}=\frac{\mu}{\sqrt{\pi}}\sqrt{\pi}=\mu \qquad \text{Therefore} \qquad \Gamma\frac{1}{2}=\sqrt{\pi}$$

$$E(X)=\mu$$

Now for median

Let by definition median of normal distribution (median=M)

$$\int_{-\infty}^{M} f(x)d(x)=\frac{1}{2}$$

$$X\to N(\mu,\sigma^2)$$

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \qquad -\infty\le X\le\infty$$

$$\int_{-\infty}^{M}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}d(x)=\frac{1}{2}$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{M}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}d(x)=\frac{1}{2}$$

Put $\quad Z=\frac{X-\mu}{\sigma}$

$$Z\sigma=x-\mu$$

$$X=Z\sigma+\mu$$

$$d(x)=\sigma\ d(z)$$

Limits When $X\to-\infty\ Then\ Z\to-\infty\quad And\quad X\to M\ \ Then\ \ Z\to\frac{M-\mu}{\sigma}$

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\frac{M-\mu}{\sigma}}e^{-\frac{1}{2}(z)^2}\sigma\ d(Z)=\frac{1}{2}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{M-\mu}{\sigma}}e^{-\frac{1}{2}(z)^2}d(Z)=\frac{1}{2} \qquad \text{(i)}$$

As we know that normal distribution is symmetrical then we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}e^{-\frac{1}{2}(z)^2}d(Z)=\frac{1}{2}=\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}(z)^2}d(Z) \qquad \text{(ii)}$$

Comparing (i) and (ii) then we get

$$\frac{M-\mu}{\sigma}=0$$

$M = \mu$

$Median = \mu$ Hence it is also equal to "μ"

Now for mode

The following two conditions are satisfied then mode is exist

$f'(x) = 0$

$f''(x) < 0$ Mean second derivative must be negative

As we know that probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$$

Differentiate with respect to "X"

$$f'(x) = \frac{d}{d(x)} f(x)$$

$$\frac{d}{d(x)} f(x) = \frac{d}{d(x)}\left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$\frac{d}{d(x)} f(x) = \frac{1}{\sigma\sqrt{2\pi}} \frac{d}{d(x)}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$\frac{d}{d(x)} f(x) = \frac{1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \frac{d}{d(x)} \frac{-1}{2}\left(\frac{X-\mu}{\sigma}\right)^2\right)$$

$$\frac{d}{d(x)} f(x) = \frac{1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \frac{-2}{2}\left(\frac{X-\mu}{\sigma}\right) \frac{d}{d(x)} \frac{X-\mu}{\sigma}\right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \left(\frac{X-\mu}{\sigma}\right) \frac{1}{\sigma} \frac{d}{d(x)}(X-\mu)\right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \left(\frac{X-\mu}{\sigma}\right) \frac{1}{\sigma}(1-0)\right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \left(\frac{X-\mu}{\sigma}\right) \frac{1}{\sigma}\right)$$

$$\frac{d}{d(x)} f(x) = \frac{-1}{\sigma^3\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} (X-\mu)\right) \qquad \text{(A)}$$

Now we equating the first derivative zero

$$0 = \frac{-1}{\sigma^3\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} (X-\mu)\right)$$

$$0 = e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} (X-\mu)$$

$$0 = (X-\mu)$$

$$X = \mu$$

Now we again differentiate eq. (A) with respect to "X"

$$f''(x) = \frac{d}{d(x)}\left(\frac{-1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} (X-\mu)\right)$$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} \frac{d}{d(x)}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} (X-\mu)\right)$$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}}\left((X-\mu)\frac{d}{d(x)} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} + e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \frac{d}{d(x)}(X-\mu)\right)$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left((X-\mu)e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\frac{d}{d(x)}\frac{-1}{2}\left(\frac{X-\mu}{\sigma}\right)^2+e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}(1-0)\right)$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left((X-\mu)e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\frac{-2}{2}\left(\frac{X-\mu}{\sigma}\right)\frac{d}{d(x)}\left(\frac{X-\mu}{\sigma}\right)+e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left(-(X-\mu)e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\frac{X-\mu}{\sigma}\right)\frac{1}{\sigma}\frac{d}{d(x)}(X-\mu)+e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left(-(X-\mu)e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\frac{X-\mu}{\sigma}\right)\frac{1}{\sigma}(1-0)+e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left(-(X-\mu)e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\frac{X-\mu}{\sigma^2}\right)+e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left(-e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\frac{X-\mu}{\sigma}\right)^2+e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$f''(x)=\frac{1}{\sigma^3\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\left(\frac{X-\mu}{\sigma}\right)^2-1\right)$$

Put $X=\mu$ then we get $f''(x)<0$ thus the $Mode=\mu$.
Hence in normal distribution
$Mean=Median=Mode=\mu$ Proved

Theorem.No.6: Show that maximum value of $f(X)$ occurs at "$X=\mu$"
Proof: As we know that

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$$

Differentiate with respect to "X"

$$f'(x)=\frac{d}{d(x)}f(x)$$

$$\frac{d}{d(x)}f(x)=\frac{d}{d(x)}\left(\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$\frac{d}{d(x)}f(x)=\frac{1}{\sigma\sqrt{2\pi}}\frac{d}{d(x)}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$\frac{d}{d(x)}f(x)=\frac{1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\frac{d}{d(x)}\frac{-1}{2}\left(\frac{X-\mu}{\sigma}\right)^2\right)$$

$$\frac{d}{d(x)}f(x)=\frac{1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\frac{-2}{2}\left(\frac{X-\mu}{\sigma}\right)\frac{d}{d(x)}\frac{X-\mu}{\sigma}\right)$$

$$\frac{d}{d(x)}f(x)=\frac{-1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\frac{X-\mu}{\sigma}\right)\frac{1}{\sigma}\frac{d}{d(x)}(X-\mu)\right)$$

$$\frac{d}{d(x)}f(x)=\frac{-1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\frac{X-\mu}{\sigma}\right)\frac{1}{\sigma}(1-0)\right)$$

$$\frac{d}{d(x)}f(x)=\frac{-1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\frac{X-\mu}{\sigma}\right)\frac{1}{\sigma}\right)$$

$$\frac{d}{d(x)}f(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}(X-\mu)\right) \qquad \text{(A)}$$

Now we equating the first derivative zero

$$0=\frac{-1}{\sigma^3\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}(X-\mu)\right)$$

$$0=e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}(X-\mu)$$

$$0=(X-\mu)$$

$$X=\mu$$

Now we see that

$$\begin{array}{llll} For & X<\mu & Then & f'(x)>0 \\ For & X>\mu & Then & f'(x)<0 \\ For & X=\mu & Then & f'(x)=0 \end{array}$$

Thus the maximum of the function $f(x)$ at $X=\mu$ *which is* $\frac{1}{\sigma\sqrt{2\pi}}$

$$\left[f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}=\frac{1}{\sigma\sqrt{2\pi}} \quad at\ \ X=\mu\right]$$

Theorem.No.7: Show that the points of inflexation in normal distribution are equidistant from mean

Proof:

As we know that probability density function

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$$

Differentiate with respect to "X"

$$f'(x)=\frac{d}{d(x)}f(x)$$

$$\frac{d}{d(x)}f(x)=\frac{d}{d(x)}\left(\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$\frac{d}{d(x)}f(x)=\frac{1}{\sigma\sqrt{2\pi}}\frac{d}{d(x)}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$\frac{d}{d(x)}f(x)=\frac{1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\frac{d}{d(x)}\frac{-1}{2}\left(\frac{X-\mu}{\sigma}\right)^2\right)$$

$$\frac{d}{d(x)}f(x)=\frac{1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\frac{-2}{2}\left(\frac{X-\mu}{\sigma}\right)\frac{d}{d(x)}\frac{X-\mu}{\sigma}\right)$$

$$\frac{d}{d(x)}f(x)=\frac{-1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\frac{X-\mu}{\sigma}\right)\frac{1}{\sigma}\frac{d}{d(x)}(X-\mu)\right)$$

$$\frac{d}{d(x)}f(x)=\frac{-1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\frac{X-\mu}{\sigma}\right)\frac{1}{\sigma}(1-0)\right)$$

$$\frac{d}{d(x)}f(x)=\frac{-1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\frac{X-\mu}{\sigma}\right)\frac{1}{\sigma}\right)$$

$$\frac{d}{d(x)}f(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}(X-\mu)\right) \qquad \text{(A)}$$

Now we equating the first derivative zero

$$0=\frac{-1}{\sigma^3\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}(X-\mu)\right)$$

$$0=e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}(X-\mu)$$

$$0=(X-\mu)$$

$$X=\mu$$

Now we again differentiate eq. (A) with respect to "X"

$$f''(x)=\frac{d}{d(x)}\left(\frac{-1}{\sigma^3\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}(X-\mu)\right)$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\frac{d}{d(x)}\left(e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}(X-\mu)\right)$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left((X-\mu)\frac{d}{d(x)}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}+e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\frac{d}{d(x)}(X-\mu)\right)$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left((X-\mu)e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\frac{d}{d(x)}\frac{-1}{2}\left(\frac{X-\mu}{\sigma}\right)^2+e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}(1-0)\right)$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left((X-\mu)e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\frac{-2}{2}\left(\frac{X-\mu}{\sigma}\right)\frac{d}{d(x)}\left(\frac{X-\mu}{\sigma}\right)+e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left(-(X-\mu)e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\frac{X-\mu}{\sigma}\right)\frac{1}{\sigma}\frac{d}{d(x)}(X-\mu)+e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left(-(X-\mu)e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\frac{X-\mu}{\sigma}\right)\frac{1}{\sigma}(1-0)+e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left(-(X-\mu)e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\frac{X-\mu}{\sigma^2}\right)+e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}\left(-e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\frac{X-\mu}{\sigma}\right)^2+e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\right)$$

$$f''(x)=\frac{1}{\sigma^3\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\left(\frac{X-\mu}{\sigma}\right)^2-1\right)$$

Now we equating the 2nd derivative zero

$$0=\frac{1}{\sigma^3\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}\left(\left(\frac{X-\mu}{\sigma}\right)^2-1\right)$$

$$0=\left(\left(\frac{X-\mu}{\sigma}\right)^2-1\right)$$

$$1=\left(\frac{X-\mu}{\sigma}\right)^2$$

$$\sigma^2=(X-\mu)^2$$

$$\sqrt{\sigma^2}=\sqrt{(X-\mu)^2}$$

$$\pm\sigma=X-\mu$$

$$X=\mu\pm\sigma$$

At $X = \mu - \sigma$ Then we get $\left[f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}} = \frac{1}{\sigma\sqrt{2\pi\ e}} \right]$

At $X = \mu + \sigma$ Then we get $\left[f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}} = \frac{1}{\sigma\sqrt{2\pi\ e}} \right]$

Hence the two points of inflection of normal curve are

$\left[(\mu - \sigma)\ ,\ \ \frac{1}{\sigma\sqrt{2\pi\ e}} \right]$ And $\left[(\mu + \sigma)\ ,\ \ \frac{1}{\sigma\sqrt{2\pi\ e}} \right]$

Theorem.No.8: Derive moment generating function of normal distribution

Proof: Let by definition of m.g.f

$$M_0(t) = E(e^{tx})$$

$$M_0(t) = \int_{-\infty}^{\infty} e^{tx} f(x) d(x)$$

$$X \to N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \qquad -\infty \le X \le \infty$$

$$M_0(t) == \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} d(x)$$

$$M_0(t) == \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} d(x)$$

Put $Z = \frac{X - \mu}{\sigma}$

$$Z\sigma = x - \mu$$

$$X = Z\sigma + \mu$$

$d(x) = Zd(z)$ And Limits remains same $-\infty \le Z \le \infty$

$$M_0(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma Z+\mu)} e^{-\frac{1}{2}(z)^2} \sigma\ d(z)$$

$$M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tZ\sigma} e^{t\mu} e^{-\frac{1}{2}(z)^2} d(z)$$

$$M_0(t) = \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Z)^2 + tZ\sigma} d(z)$$

$$M_0(t) = \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(Z^2 - 2tZ\sigma\right)} d(z)$$

$$M_0(t) = \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(Z^2 - tZ\sigma + (\sigma t)^2 - (\sigma t)^2\right)} d(z)$$

$$M_0(t) = \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(Z^2 - 2tZ\sigma + (\sigma t)^2\right)} e^{-\frac{1}{2}\left(-(\sigma t)^2\right)} d(z)$$

$$M_0(t) = \frac{e^{t\mu} e^{\frac{1}{2}(\sigma t)^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Z - \sigma t)^2} d(z)$$

$$M_0(t) = \frac{e^{\mu t + \frac{1}{2}(\sigma t)^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Z - \sigma t)^2} d(z)$$

Put $w = z - t\sigma$

$dw = dz$

Limits remain same

$$M_0(t)=\frac{e^{\mu t+\frac{1}{2}(\sigma t)^2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}w^2}d(w)$$

As $f(w)=e^{-\frac{1}{2}w^2}$

Put $w=-w$

$f(-w)=e^{-\frac{1}{2}(-w)^2}=e^{-\frac{1}{2}w^2}=f(w)$ Its even function then we gets

$$M_0(t)=\frac{2e^{\mu t+\frac{1}{2}(\sigma t)^2}}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-\frac{1}{2}w^2}d(w)$$

Put $y=\frac{1}{2}w^2$

$\sqrt{2y}=w$

$\frac{1}{2}(2y)^{-\frac{1}{2}}2d(y)=d(w)$

$(2y)^{-\frac{1}{2}}d(y)=d(w)$ And Limits remains same $0\le y\le\infty$

$$M_0(t)=\frac{2e^{\mu t+\frac{1}{2}(\sigma t)^2}}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-y}(2y)^{-\frac{1}{2}}d(y)$$

$$M_0(t)=\frac{2e^{\mu t+\frac{1}{2}(\sigma t)^2}}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{y^{\frac{1}{2}-1}}{2^{\frac{1}{2}}}e^{-y}d(y)$$

$$M_0(t)=\frac{2e^{\mu t+\frac{1}{2}(\sigma t)^2}}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{y^{\frac{1}{2}-1}}{\sqrt{2}}e^{-y}d(y)$$

$$M_0(t)=\frac{2e^{\mu t+\frac{1}{2}(\sigma t)^2}}{2\sqrt{\pi}}\int_{0}^{\infty}y^{\frac{1}{2}-1}e^{-y}d(y)$$

$$M_0(t)=\frac{e^{\mu t+\frac{1}{2}(\sigma t)^2}}{\sqrt{\pi}}\int_{0}^{\infty}y^{\frac{1}{2}-1}e^{-y}d(y)\qquad\text{(i)}$$

As we know that Gamma function is

$$\Gamma\alpha\,\beta^{\alpha}=\int_{0}^{\infty}y^{\alpha-1}e^{-y/\beta}\ d(y)\qquad\text{(ii)}$$

Comparing (i) and (v) then we get

$\alpha=\frac{1}{2}$ *And* $\beta=1$ Then eq. (ii) becomes

$$M_0(t)=\frac{e^{\mu t+\frac{1}{2}(\sigma t)^2}}{\sqrt{\pi}}\Gamma\frac{1}{2}\,1^{\frac{1}{2}}$$

$$M_0(t)=\frac{e^{\mu t+\frac{1}{2}(\sigma t)^2}}{\sqrt{\pi}}\sqrt{\pi}$$ Therefore $\Gamma\frac{1}{2}=\sqrt{\pi}$

$M_0(t)=e^{\mu t+\frac{1}{2}(\sigma t)^2}$ Required result

Theorem.No.9: Derive moment generating function of standardized normal distribution

Proof: Let by definition of m.g.f

$M_0(t)=E(e^{tZ})$

$$M_0(t)=\int_{-\infty}^{\infty}e^{tz}f(z)d(z)$$

$z\to N(0,1)$

$f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z)^2}$ $-\infty\le Z\le\infty$

$$M_0(t) == \int_{-\infty}^{\infty} e^{tZ} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(Z)^2} d(Z)$$

$$M_0(t) == \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tZ} e^{-\frac{1}{2}(Z)^2} d(Z)$$

$$M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Z)^2 + tZ} d(z)$$

$$M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(Z^2 - 2t\right)} d(z)$$

$$M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(Z^2 - tZ + (t)^2 - (t)^2\right)} d(z)$$

$$M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(Z^2 - 2tZ + (t)^2\right)} e^{-\frac{1}{2}\left(-(t)^2\right)} d(z)$$

$$M_0(t) = \frac{e^{\frac{1}{2}(t)^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Z-t)^2} d(z)$$

$$M_0(t) = \frac{e^{\mu t + \frac{1}{2}(\sigma t)^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Z-t)^2} d(z)$$

Put $w = z - t$

$dw = dz$

Limits remain same

$$M_0(t) = \frac{e^{\frac{1}{2}(t)^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2} d(w)$$

As $f(w) = e^{-\frac{1}{2}w^2}$

Put $w = -w$

$f(-w) = e^{-\frac{1}{2}(-w)^2} = e^{-\frac{1}{2}w^2} = f(w)$ Its even function then we gets

$$M_0(t) = \frac{2e^{\frac{1}{2}(t)^2}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}w^2} d(w)$$

Put $y = \frac{1}{2}w^2$

$\sqrt{2y} = w$

$\frac{1}{2}(2y)^{-\frac{1}{2}} 2d(y) = d(w)$

$(2y)^{-\frac{1}{2}} d(y) = d(w)$ And Limits remains same $0 \le y \le \infty$

$$M_0(t) = \frac{2e^{\frac{1}{2}(t)^2}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-y} (2y)^{-\frac{1}{2}} d(y)$$

$$M_0(t) = \frac{2e^{\frac{1}{2}t^2}}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{y^{\frac{1}{2}-1}}{2^{\frac{1}{2}}} e^{-y} d(y)$$

$$M_0(t) = \frac{2e^{\frac{1}{2}t^2}}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{y^{\frac{1}{2}-1}}{\sqrt{2}} e^{-y} d(y)$$

$$M_0(t) = \frac{e^{\frac{1}{2}t^2}}{\sqrt{\pi}} \int_{0}^{\infty} y^{\frac{1}{2}-1} e^{-y} d(y)$$

$$M_0(t) = \frac{e^{\frac{1}{2}t^2}}{\sqrt{\pi}} \int_{0}^{\infty} y^{\frac{1}{2}-1} e^{-y} d(y) \qquad \text{(i)}$$

As we know that Gamma function is

$$\overline{)\alpha}\beta^{\alpha}=\int_{0}^{\infty} y^{\alpha-1}e^{-y/\beta}\ d(y) \qquad \text{(ii)}$$

Comparing (i) and (v) then we get

$\alpha=\dfrac{1}{2}$ $\quad And \quad \beta=1$ Then eq. (ii) becomes

$$M_0(t)=\frac{e^{\frac{1}{2}t^2}}{\sqrt{\pi}}\overline{)\frac{1}{2}}1^{\frac{1}{2}}$$

$$M_0(t)=\frac{e^{\frac{1}{2}t^2}}{\sqrt{\pi}}\sqrt{\pi}$$ Therefore $\overline{)\frac{1}{2}}=\sqrt{\pi}$

$$M_0(t)=e^{\frac{1}{2}t^2}$$ Required result

Theorem.No.10: Show those odd order moments of a normal distribution are equal to zero

Proof: Let by definition

$$\mu_{2n+1}=E[X-\mu]^{2n+1}=\int_{-\infty}^{\infty}[X-\mu]^{2n+1}f(x)d(x)$$

$$X\to N(\mu,\sigma^2)$$

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \qquad -\infty\le X\le\infty$$

$$\mu_{2n+1}=\int_{-\infty}^{\infty}[X-\mu]^{2n+1}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}d(x)$$

$$\mu_{2n+1}=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}[X-\mu]^{2n+1}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}d(x)$$

Put $\quad Z=\dfrac{X-\mu}{\sigma}$

$$Z\sigma=x-\mu$$

$$X=Z\sigma+\mu$$

$d(x)=\sigma\ d(z)$ And Limits remains same $-\infty\le Z\le\infty$

$$\mu_{2n+1}=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}[\sigma Z]^{2n+1}e^{-\frac{1}{2}(Z)^2}\sigma d(Z)$$

$$\mu_{2n+1}=\frac{[\sigma]^{2n+1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[Z]^{2n+1}e^{-\frac{1}{2}(Z)^2}d(Z) \qquad \text{(i)}$$

As $f(z)=Z^{2n+1f}e^{-\frac{1}{2}Z^2}$

Put $Z=-Z$

$f(-Z)=-f(Z)$

It is odd function then eq. (i) becomes

$$\mu_{2n+1}=\frac{[\sigma]^{2n+1}}{\sqrt{2\pi}}(0)=0$$ Hence proved

Theorem.No.11: Show that Mean deviation of normal distribution is
$"Q.D=\frac{2}{3}\sigma=0.67\sigma"$

Proof: As we know that $Q.D=\frac{Q_3-Q_1}{2}$ (A)

First we find $"Q_3 \ \ And \ \ Q_1"$

Let "Q_1" be a point that contain 25% area below it. Then we get the ordinate from μ to Q_1 is 0.5-0.25=0.25

By S.N.V

$Z=\frac{X-\mu}{\sigma}$

$P(Z/P=0.250)=\frac{Q_1-\mu}{\sigma}$ Using area table inversely

$0.67=\frac{X-\mu}{\sigma}$ Sign is negative because left sided

$-0.67\sigma=Q_1-\mu$

$Q_1=\mu-0.67\sigma$

Let "Q_3" be a point that contain 75% area below it. Then we get ordinate from Q_3 to μ is 0.7-0.5=0.25

By S.N.V

$Z=\frac{X-\mu}{\sigma}$

$P(Z/P=0.250)=\frac{Q_3-\mu}{\sigma}$ Using area table inversely

$0.67=\frac{Q_3-\mu}{\sigma}$ Sign is positive because right sided

$0.67\sigma=Q_3-\mu$

$Q_3=\mu+0.67\sigma$

Substitute the values in (A) we get

$Q.D=\frac{\mu+0.67\sigma-(\mu-0.67\sigma)}{2}$

$Q.D=\frac{\mu+0.67\sigma-\mu+0.67\sigma}{2}=\frac{\mu 0.67\sigma+0.67\sigma}{2}=\frac{2(0.67\sigma)}{2}=0.67\sigma=\frac{2}{3}\sigma$ Proved

Importance of normal distribution

There are some importance of normal distribution are given below

i) Normal distribution is called probability distribution for errors of measurements.

ii) It is used in solving problems both in probability and in statistical inference.

iii) Many natural facts follow normal distribution.

iv) This distribution helps for drawing conclusions about population on the basis of sample information.

v) This distribution is used in many other subjects

vi) Many other distribution derived from normal distribution

vii) Shape of normal curve determined by its parameters ($\mu \, and \, \sigma^2$)

a) μ Changes the position of normal curve along horizontal axis

b) σ^2 Indicates horizontal spread

viii) The graph of normal distribution is bell-shaped and symmetrical

ix) Normal distribution has two parameters μ and σ^2. It is denoted by $X \to N(\mu,\sigma^2)$

X) It is very important in applied statistics

Binomial approximation to normal

When total number of trials "n is so much large ($n \geq 30$) and $p \equiv q=\frac{1}{2}$ binomial probability tends to a continuous distribution known as normal distribution

i.e. $\lim_{n\to\infty} b(x;n,p) \to N(\mu,\sigma^2)$

Poisson approximation to normal distribution

When μ the mean of Poisson distribution is so much large (as it approaches to ∞) then Poisson distribution tends to a continuous distribution known as normal distribution

$\lim_{t\to\infty} P(x;\mu) = N(\mu,\sigma^2)$

Asymptotic curve

The normal curve approaches but never touches the base line (horizontal axis). So the curve is asymptotic to the horizontal axis as $X \to \pm\infty$

What do you mean by continuity correction?

Continuity correction: Binomial probability distribution is discrete which is the probability for a specified value "X" and the normal distribution is continuous, which is probability for an interval. Therefore, using binomial approximation to normal , a discrete value of binomial variable "X" is to be replaced by an interval from "X-0.5 " to "X+0.5" , before the "Z" values are computed. This sort of adjustment is called continuity correction.

For example:

i) $P(X \ge 240) = P(X \ge 239.5)$

ii) $P(X > 240) = P(X > 240.5)$

iii) $P(X \le 240) = P(X \le 240.5)$

iv) $P(X < 240) = P(X < 239.5)$

v) $P(240 \le X \le 260) = P(239.5 \le X \le 260.5)$

vi) $P(240 < X < 260) = P(240.5 < X < 259.5)$

vii) $P(X = 240) = P(239.5 \le X \le 240.5)$

Example: If the random variable "Z" has the standard normal distribution, find;

i) $P(Z < -0.46)$ ii) $P(Z > 0.46)$ iii) $P(Z < 1.46)$

iv) $P(Z > -1.46)$ v) $P(Z > 1.46)$ vi) $P(Z < -1.48)$

vii) $P(Z > -1.96)$ viii) $P(Z < 0)$ ix) $P(Z > 0)$

x) $P(0 < Z < 1.15)$ xi) $P(-1.25 < Z < 0)$ xii) $P(0.65 < Z < 1.99)$

xiii) $P(-1.65 < Z < -1.32)$ xiv) $P(-0.65 < Z < 1.99)$ xv) $P(Z > mean)$

xvi) $P(Z > Vaiance)$

Solution:

i) $P(Z < -0.46) = ?$

$= p(-\infty < Z < 0) - P(-0.46 < Z < 0)$

= 0.5-0.1772=0.3228

ii) $P(Z > 0.46) = ?$

$= p(o < Z < \infty) - P(0 < Z < 0.46)$

= 0.5-0.1772=0.3228

iii) $P(Z < 1.46) = ?$

$= p(-\infty < Z < 0) + P(0 < Z < 1.46)$

= 0.5+0.4279=0.9279

iv) $P(Z > -1.46) = ?$

$= p(0 < Z < \infty) + P(-1.46 < Z < 0)$

= 0.5+0.4279=0.9279

v) $P(Z > 1.46) = ?$

$= p(0 < Z < \infty) - P(0 < Z < 1.46)$

= 0.5-0.4279=0.0721

vi) $P(Z < -1.48) = ?$

$= p(-\infty < Z < 0) - P(-1.48 < Z < 0)$

= 0.5-0.4306=0.0694

vii) $P(Z > -1.96) = ?$

$= p(0 < Z < \infty) + P(-1.96 < Z < 0)$

= 0.5+0.4750=0.9750

viii) $P(Z<0)=?$
$=p(-\infty<Z<0)-P(0<Z<o)$
= 0.5-0=0.5

ix) $P(Z>0)=?$
$=p(0<Z<\infty)-P(0<Z<0)$
= 0.5-0=0.5

x) $P(0<Z<1.15)=?$
$=p(0<Z<1.15)-P(0<Z<0)$
= 0.3749-0=0.3749

xi) $P(-1.25<Z<0)=?$
$=p(-1.25<Z<0)-P(0<Z<0)$
= 0.3944-0=0.3944

xii) $P(0.65<Z<1.99)=?$
$=p(0<Z<1.99)-P(0<Z<0.65)$
= 0.4767-0.2422=0.2345

xiii) $P(-1.65<Z<-1.32)=?$
$=p(-1.65<Z<0)-P(-1.32<Z<0)$
= 0.4505-0.4066=0.0439

xiv) $P(-0.65<Z<1.99)=?$
$=p(0<Z<1.99)+P(-0.65<Z<0)$
= 0.4767+0.2422=0.7189

xv) $P(Z>mean)=p(Z>0)=?$
$=P(0<Z<\infty)-P(0<Z<0)$
= 0.5-0=0.5

xvi) $P(Z>Variance)=p(Z>1)=?$
$=P(0<Z<\infty)-P(0<Z<1.0)$
= 0.5-0.3413=0.1587

xvii) $P(-1.11>Z>1.22)=?$
$=1-\left[P(-1.11<Z<0)+P(0<Z<1.22)\right]$
=1-(0.3665+0.3888) =0.2447

Alternative way
$P(-1.11>Z>1.22)=P(Z<-1.11)+P(Z>1.22)$ ------------------------------------ (A)
First we find
$P(Z<-1.11)=?$

$=P(-\infty<Z<0)-P(-1.11<Z<0)$
=0.5-0.3665=0.1335
$P(Z>1.22)=?$

$=P(0<Z<\infty)-P(0<Z<1.22)$
=0.5-0.3888=0.1112 then equation (A) becomes
$P(-1.11>Z>1.22)=P(Z<-1.11)+P(Z>1.22)$ =0.1335+0.1112=0.2447
Example: In a normal distribution M.D=3.9895 then find standard deviation, Quartile deviation, Second and fourth moment about mean of the normal distribution.

Solution:

i) As we know that $M.D = 0.7979\sigma$ in a normal distribution. So, we get

$3.9895 = 0.7979\sigma$ Because M.D=3.9895

$$\sigma = \frac{3.9895}{0.7979} = 5.0 = Standard\ deviaton$$

ii) As we know that $Q.D = 0.6745\sigma$ in a normal distribution. So, we get

$Q.D = 0.6745(5) = 3.3725 = Quartile\ deviation$

iii) Second moment about mean $\mu_2 = \sigma^2 = 5^2 = 25$

iv) Fourth moment about mean $\mu_4 = 3\sigma^4 = 3(5)^4 = 1875$

vii) Between 15 and 30 viii) greater than 50 ix) Find the area not less than

Example: In a normal distribution with $\mu = 20$ and standard deviation $\sigma = 5$.

Find the area

i) Less than 15 ii) More than 25 iii) Less than 25

iv) More than 15 v) Between 15 and 18 vi) Between 25 and 30

15 0r more than 25 $(15 < or > 25)$

Solution: Given that $\mu = 20$ and $\sigma = 5$.

Standardized normal variate

$$Z = \frac{X - \mu}{\sigma}$$

i) $P(\text{Less than } 15) = P(X < 15) = ?$

X=15 $\mu = 20$ and $\sigma = 5$.

By S.N.V

$$Z = \frac{X - \mu}{\sigma} = \frac{15 - 20}{5} = \frac{-5}{5} = -1$$

Then we get

$$P(X < 15) = P(Z < -1.0) = P(-\infty < Z < 0) - P(-1.0 < Z < 0) = 0.5 - 0.3413 = 0.1587$$

ii) $P(More\ \text{than } 25) = P(X > 25) = ?$

X=25 $\mu = 20$ and $\sigma = 5$.

By S.N.V

$$Z = \frac{X - \mu}{\sigma} = \frac{25 - 20}{5} = \frac{5}{5} = 1.0$$

Then we get

$$P(X > 25) = P(Z > 1.0) = P(0 < Z < \infty) - P(0 < Z < 1.0) = 0.5 - 0.3413 = 0.1587$$

iii) Less than 25

$P(Less\ \text{than } 25) = P(X < 25) = ?$

X=25 $\mu = 20$ and $\sigma = 5$.

By S.N.V

$$Z = \frac{X - \mu}{\sigma} = \frac{25 - 20}{5} = \frac{5}{5} = 1.0$$

Then we get

$$P(X < 25) = P(Z < 1.0) = P(-\infty < Z < 0) + P(0 < Z < 1.0) = 0.5 + 0.3413 = 0.8413$$

iv) $P(More\ \text{than } 15) = P(X > 15) = ?$

X=15 $\mu = 20$ and $\sigma = 5$.

By S.N.V

$$Z = \frac{X - \mu}{\sigma} = \frac{15 - 20}{5} = \frac{-5}{5} = -1$$

Then we get

$$P(X > 15) = P(Z > -1.0) = P(0 < Z < \infty) + P(-1.0 < Z < 0) = 0.5 + 0.3413 = 0.8413$$

v) Between 15 and 18

$P(\text{Between } 15 \text{ and } 18) = P(15 < X < 18) = ?$

$X_1=15$ $\mu = 20$ and $\sigma = 5$.

By S.N.V

$$Z_1 = \frac{X_1 - \mu}{\sigma} = \frac{15-20}{5} = \frac{-5}{5} = -1.0$$

$X_2=18$ $\mu = 20$ and $\sigma = 5$.

By S.N.V

$$Z_2 = \frac{X_2 - \mu}{\sigma} = \frac{18-20}{5} = \frac{-2}{5} = -0.4$$

Then we get

$P(15 < X < 18) = P(-1.0 < Z > -0.4) = P(-0.4 < Z < 0) - P(-1.0 < Z < 0) = 0.3413 - 0.1554 = 0.1859$

vi) Between 25 and 30

$P(\text{Between } 25 \text{ and } 30) = P(25 < X < 30) = ?$

$X_1=25$ $\mu = 20$ and $\sigma = 5$.

By S.N.V

$$Z_1 = \frac{X_1 - \mu}{\sigma} = \frac{25-20}{5} = \frac{5}{5} = 1.0$$

$X_2=18$ $\mu = 20$ and $\sigma = 5$.

Then we get

$P(25 < X < 30) = P(1.0 < Z > 2.0) = P(0 < Z < 2.0) - P(0 < Z < 1.0) = 0.4772 - 0.3413 = 0.1359$

vii) Between 15 and 30

$P(\text{Between } 15 \text{ and } 30) = P(15 < X < 30) = ?$

$X_1=15$ $\mu = 20$ and $\sigma = 5$.

By S.N.V

$$Z_1 = \frac{X_1 - \mu}{\sigma} = \frac{15-20}{5} = \frac{-5}{5} = -1.0$$

$X_2=30$ $\mu = 20$ and $\sigma = 5$.

By S.N.V

$$Z_2 = \frac{X_2 - \mu}{\sigma} = \frac{30-20}{5} = \frac{10}{5} = 2.0$$

Then we get

$P(15 < X < 30) = P(-1.0 < Z < 2.0) = P(-1.0 < Z < 0) + P(0 < Z < 2.0) = 0.3413 + 0.4772 = 0.8185$

viii) Greater than 50

$P(Greater \text{ than } 50) = P(X > 50) = ?$

$X=50$ $\mu = 20$ and $\sigma = 5$.

By S.N.V

$$Z = \frac{X - \mu}{\sigma} = \frac{50-20}{5} = \frac{30}{5} = 6.0$$

Then we get

$P(X > 50) = P(Z > 6.0) = P(0 < Z < \infty) - P(0 < Z < 6.0) = 0.5 - 0.5 = 0$

ix) Find the area not less than 15 0r more than 25 $(15 < or > 25)$

$P(less \text{ than } 15) = P(X < 15) = ?$

$X=15$ $\mu = 20$ and $\sigma = 5$.

By S.N.V

$$Z = \frac{X - \mu}{\sigma} = \frac{15-20}{5} = \frac{-5}{5} = -1.0$$

Then we get

$P(X < 15) = P(Z < -1.0) = P(-\infty < Z < 0) - P(-1.0 < Z < 0) = 0.5 - 0.3413 = 0.1587$

$P(more \text{ than } 25) = P(X > 25) = ?$

$X=25$ $\mu = 20$ and $\sigma = 5$.

By S.N.V

$$Z = \frac{X-\mu}{\sigma} = \frac{25-20}{5} = \frac{5}{5} = 1.0$$

Then we get

$\mathrm{P(X > 25) = P(Z > 1.0)} = P(0 < Z < \infty) - P(0 < Z < 1.0) = 0.5 - 0.3413 = 0.1587$

And required result

$P(15 < or > 25) = P(X < 15) + P(X > 25) = 0.1587 + 0.1587 = 0.3174$

Example: In a normal distribution the mean is five and the variance is one. Write down its equation. Also find the value of maximum ordinate correct to two places decimals.

Solution: As $X \to N(5,1)$

$\mu = 5$ And $\sigma^2 = 1.0$

The equation of normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \qquad -\infty \le X \le +\infty$$

When $\mu = 5$ And $\sigma^2 = 1.0$ then it becomes

$$f(x) = \frac{1}{1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-5}{1}\right)^2}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(X-5)^2} \quad \text{Required result} \qquad -\infty \le X \le +\infty$$

ii) The equation of normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \qquad \text{Maximum ordinate at } X = \mu = 5$$

$$f(x) = \frac{1}{1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{5-5}{1}\right)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(0)} = \frac{1}{\sqrt{2\pi}} e^{0} = \frac{1}{1\sqrt{2\pi}} = 0.3989 = 0.40$$

Note: When ask the question find maximum ordinate then put the value of mean and variable "X" as same value.

Example: The scores made by candidates in a certain test are normally distributed with mean 500 and standard deviation 100. What percent of the candidates received scores?

Find the probability which differ from mean by more than 150?

Solution: Given that $\mu = 500$ And $\sigma = 100$

Mean differ by more than 150 means that we add and subtract 150 from mean then we get $\mathrm{P(350 \le X \ge 650)} = P(X \le 350) + P(X \ge 650) = ?$

$\mathrm{P(X \le 350)} = ?$

$P(more\ \mathrm{than\ 25) = P(X > 25)} = ?$

X=350 $\mu = 400$ and $\sigma = 100$.

By S.N.V

$$Z = \frac{X-\mu}{\sigma} = \frac{350-500}{100} = \frac{-150}{100} = -1.50$$

Then we get

$\mathrm{P(X \le 350) = P(Z < -1.50)} = P(-\infty < Z < 0) - P(-1.50 < Z < 0) = 0.5 - 0.4332 = 0.0.0668$

$\mathrm{P(X \ge 650)} = ?$

X=650 $\mu = 400$ and $\sigma = 100$.

By S.N.V

$$Z = \frac{X-\mu}{\sigma} = \frac{650-500}{100} = \frac{150}{100} = 1.50$$

Then we get

$\mathrm{P(X \ge 650) = P(Z > 1.50)} = P(0 < Z < \infty) - P(0 < Z < 1.50) = 0.5 - 0.4332 = 0.0.0668$

And required result

$P(Mean\, differ\, more\, than 150) = P(X \le 350) + P(X \ge 650) = 0.0668 + 0.0668 = 0.1336$
Hence required percentage 13.36%.

Example: In a certain examination 3000 students appeared. The average marks obtained were 50% and standard deviation was 5%. How many students do you expect who obtain:

i) More than 60% marks ii) Less than 40% marks iii) Between 40% and 60% marks

Solution: Given that $\mu = 0.5$ $\sigma = 0.05$ And $N = 3000$

i) More than 60% marks

$P(More \text{ than } 0.60) = \mathrm{P(X > 0.60)} = ?$

X=0.60 $\mu = 0.5$ $\sigma = 0.05$ And $N = 3000$

By S.N.V

$Z = \dfrac{X-\mu}{\sigma} = \dfrac{0.60-0.50}{0.05} = \dfrac{0.10}{0.05} = 2.0$ Then we get

$\mathrm{P(X > 0.60) = P(Z > 2.0)} = P(0 < Z < \infty) - P(0 < Z < 2.0) = 0.5 - 0.47725 = 0.0228$

Hence required expected no. of students obtained more than 60% marks is 68

ii) Less than 40% marks

$P(Less \text{ than } 0.40) = \mathrm{P(X < 0.40)} = ?$

X=0.40 $\mu = 0.5$ $\sigma = 0.05$ And $N = 3000$

By S.N.V

$Z = \dfrac{X-\mu}{\sigma} = \dfrac{0.40-0.50}{0.05} = \dfrac{-0.10}{0.05} = -2.0$

Then we get

$\mathrm{P(X < 0.40) = P(Z < -2.0)} = P(-\infty < Z < 0) - P(-2.0 < Z < 0) = 0.5 - 0.47725 = 0.0228$

Hence required expected no. of students obtained less than 40% marks is 68

iii) Between 40% and 60% marks

$P(0.40 < X < 0.60)) = ?$

X_1=0.40 $\mu = 0.5$ $\sigma = 0.05$

By S.N.V

$Z_1 = \dfrac{X_1-\mu}{\sigma} = \dfrac{0.40-0.50}{0.05} = \dfrac{-0.10}{0.05} = -2.0$

X_2=0.60

By S.N.V

$Z_2 = \dfrac{X_2-\mu}{\sigma} = \dfrac{0.60-0.50}{0.05} = \dfrac{0.10}{0.05} = 2.0$

$P(0.40 < X < 0.60)) = P(-2.0 < Z < 2.0) = \mathrm{P(-2.0 < Z < 0) + P(0 < Z < 2.0)}$

$= 0.47725 + 0.47725 = 0.9545$

Hence required expected no. of students obtained between 40% and 60% marks is 2864

Example: A random variable "X" is normally distributed with mean=70 and S.D=5.

i) Find a point that has 87.9% of the distribution below it.

ii) Find a point that has 81.7% of the distribution above it.

iii) Find two such points between which the central 70% of the distribution lies.

iii) Find two such points between which the central 90% of the distribution lies.

Solution: Given that $\mu = 70$ $\sigma = 5$

i) Find a point that has 87.9% of the distribution below it.

Let "X" be a point that contain ($87.9\% = (\dfrac{87.9}{100}) = 0.879$) area below the point. Then the area between the point mean=40 and "X" is (0.0.879-0.50=0.379)

By S.N.V

$Z = \dfrac{X-70}{5}$

$P(Z / P = 0.379) = \dfrac{X-70}{5}$ Using area table inversely

$1.17 = \dfrac{X-70}{5}$ Sign is positive because right sided

$5(1.17) = X - 70$
$5.85 + 70 = X$
$X = 75.85$ Required result
ii) Find a point that has 81.7% of the distribution above it.
Let "X" be a point that contain ($81.7\% = (\frac{81.7}{100}) = 0.817$) area below the point. Then the area between the point mean=40 and "X" is (0.0.817-0.50=0.317)
By S.N.V
$$Z = \frac{X - 70}{5}$$

$P(Z / P = 0.317) = \frac{X - 70}{5}$ Using area table inversely

$-0.90 = \frac{X - 70}{5}$ Sign is negative because left sided

$5(-0.90) = X - 70$
$-4.5 + 70 = X$
$X = 65.50$ Required result

iii) Let "X_1 and X_2" be the two points that contain 70% area. As we know that normal distribution is symmetrical, so we divide equally $\left(\frac{0.70}{2}\right) = 0.350$ either side of mean.

By S.N.V

$Z = \frac{X - 70}{5}$ Using area table inversely

$$P(Z_1 / P = 0.35) = \frac{X_1 - 70}{5}$$

$-1.04 = \frac{X_1 - 70}{5}$ Sign is negative because left sided

$5(-1.04) = X_1 - 70$

$-5.20 = X_1 - 70$

$X_1 = -5.20 + 70 = 64.80$

Similarly for "X_2"

$$P(Z_2 / P = 0.35) = \frac{X_2 - 70}{5}$$

$1.04 = \frac{X_2 - 70}{5}$ Sign is positive because right sided

$5(1.04) = X_2 - 70$

$5.20 = X_2 - 70$

$X_2 = 5.20 + 70 = 75.20$

Hence the required points are (64.80, 75.20)

iv) Let "X_1 and X_2" be the two points that contain 90% area. As we know that normal distribution is symmetrical, so we divide equally $\left(\frac{0.90}{2}\right) = 0.450$ either side of mean.

By S.N.V

$Z = \frac{X-70}{5}$ Using area table inversely

$P(Z_1 / P = 0.45) = \frac{X_1 - 70}{5}$

$-1.645 = \frac{X_1 - 70}{5}$ Sign is negative because left sided

$5(-1.645) = X_1 - 70$

$-8.225 = X_1 - 70$

$X_1 = -8.8225 + 70 = 61.775$

Similarly for "X_2"

$P(Z_2 / P = 0.45) = \frac{X_2 - 70}{5}$

$1.645 = \frac{X_2 - 70}{5}$ Sign is positive because right sided

$5(1.645) = X_2 - 70$

$8.225 = X_2 - 70$

$X_2 = 8.8225 + 70 = 78.225$

Hence the required points are (61.775, 78.225)

Example: In a normal distribution $\mu = 40$ and $\sigma = 3.8$. Find i) Two points such that the curve has a 98% chance of falling between

Solution: As $X \to N(40, 14.44)$

$\mu = 40$ $\sigma^2 = 14.44$ And $\sigma = 3.8$

i) Let "X_1 and X_2" be the two points that contain 98% area. As we know that normal distribution is symmetrical, so we divide equally $\left(\frac{0.98}{2}\right) = 0.490$ either side of mean.

From the area table the value of Z for which the area between 0(zero) and Z=0.490 is 2.33. Therefore "X_1" lie on the left side so sign of Z_1 is negative and Z_2 is positive because it is tight sided of the mean.

By S.N.V

$Z = \frac{X - \mu}{\sigma}$ $P(Z_1 / P = 0.490) = \frac{X_1 - 40}{3.8}$ Using area table inversely

$-2.33 = \frac{X_1 - 40}{3.8}$

$3.8(-2.33) = X_1 - 40$

$-8.854 = X_1 - 40$

$X_1 = -8.854 + 40 = 31.146$

Similarly for "X_2"

$$P(Z_2 / P = 0.490) = \frac{X_2 - 40}{3.8}$$

$$2.33 = \frac{X_2 - 40}{3.8}$$

$$3.8(2.33) = X_2 - 40$$

$$8.854 = X_2 - 40$$

$$X_2 = 8.854 + 40 = 48.854$$

Hence the required points are (31.146, 48.854)

Example: If X is N (24, 16) then find the i) 33rd percentile ii) 9th deciles iii) 90th percentiles iv) 60th percentiles.

Solution: Given that $\mu = 24$ $\sigma^2 = 16$ $\sigma = 4$

i) 33rd percentile=?

Let "P_{33}" be a point contain ($33\% = (\frac{33}{100}) = 0.33$) values below the point. Then the area between the point P_{33} and mean=24 is (0.5-033=0.17) And the of Z corresponding the area 0.17 is 0.44

$$P(Z / P = 0.17) = \frac{P_{33} - 24}{4}$$ Using area table inversely

$$-0.44 = \frac{P_{33} - 24}{4}$$ Sign is negative because left sided

$$4(-0.44) = P_{33} - 24$$

$$-1.76 = P_{33} - 24$$

$-1.76 + 24 = P_{33}$ And $P_{33} = 22.24$ Required result

ii) 9th deciles=?

Let "D_9=P_{90}" be a point contain $(90\% = \frac{9}{10} = 0.90)$ values below the point. Then the area between the point"D_9" and mean=24 is (0.9-0.5=0.40) And the value of Z corresponding the area 0.40 is 1.28

$$P(Z / P = 0.40) = \frac{D_9 - 24}{4}$$ Using area table inversely

$$1.28 = \frac{D_9 - 24}{4}$$ Sign is positive because right sided

$$4(1.28) = D_9 - 24$$

$$5.12 = D_9 - 24$$

$$5.12 + 24 = D_9$$

$D_9 = 29.12$ Required result

iii) 10th percentiles be a point contain ($10\% = (\frac{10}{100}) = 0.10$) values below the point. Then the area between the point P_{10} and mean=24 is (0.5-0.10=0.40) and the of Z corresponding the area 0.40 is 0.1.28

$$P(Z / P = 0.40) = \frac{P_{10} - 24}{4}$$ Using area table inversely

$$-1.28 = \frac{P_{10} - 24}{4}$$ Sign is negative because left sided

$$4(-1.28) = P_{10} - 24$$

$$-5.12 = P_{10} - 24$$
$$-5.12 + 24 = P_{10}$$
$P_{33} = 18.88$ Required result

iv) P_{60} be a point that contain ($60\% = (\frac{60}{100}) = 0.60$) values below the point. Then the area between the point P_{10} and mean=24 is (0.60-0.50=0.10) and the of Z corresponding the area 0.10 is 0.25

$$P(Z / P = 0.10) = \frac{P_{60} - 24}{4}$$ Using area table inversely

$$0.25 = \frac{P_{10} - 24}{4}$$ Sign is positive because right sided

$$4(0.25) = P_{60} - 24$$

$$1.0 = P_{60} - 24$$
$1.0 + 24 = P_{60}$ And $P_{60} = 25.0$ Required result

Example: Given a normal distribution with $\mu = 40$ and $\sigma = 6$, find the value of "X" that has; i) 38% of the area below it. ii) 5% of the area above it.

Solution: Given that $\mu = 40$ and $\sigma = 6$

i) Let "X" be a point that contain ($38\% = (\frac{38}{100}) = 0.38$) area below the point. Then the area between the point "X" and mean=40 is (0.50-0.38=0.12) and the of Z corresponding the area 0.12 is 0.31

$$P(Z / P = 0.12) = \frac{X - 40}{6}$$ Using area table inversely

$$-0.31 = \frac{X - 40}{6}$$ Sign is negative because left sided

$$6(-0.31) = X - 40$$

$$-1.86 = X - 40$$
$$-1.86 + 40 = X$$
$X = 38.14$ Required result

ii) Let "X" be a point that contain ($5\% = (\frac{5}{100}) = 0.05$) area above the point. Then the area between the mean=40 and point "X" is (0.50-0.05=0.45) and the value of Z corresponding the area 0.45 is 1.645

$$P(Z / P = 0.45) = \frac{X - 40}{6}$$ Using area table inversely

$$1.645 = \frac{X - 40}{6}$$ Sign is positive because right sided

$$6(1.645) = X - 40$$

$$9.870 = X - 40$$
$$9.870 + 40 = X$$
$X = 49.87$ Required result

Example: In a normal distribution 7% of the items are under 35 and 89% are under 63. What is the mean and standard deviation of the distribution?
Solution: As we know that 7% of the items are under 35 mean that $P(X<35)=7\%=(\frac{7}{100})=0.07$

And that 7% of the items are under 35 mean that $P(X<63)=89\%=(\frac{89}{100})=0.89$

Now

$$P(X<35)=7\%=(\frac{7}{100})=0.07$$

The 7% items are under 35, the area to the left of the ordinate "X=35" and mean is (0.5-0.07=0.43). Then Z_1 corresponding to the area 0.43 is 1.48

Using area table inversely

$$P(Z_1/P=0.43)=\frac{35-\mu}{\sigma}$$

$-1.48=\frac{35-\mu}{\sigma}$ Sign is negative because left sided

$$-1.48\sigma=35-\mu$$

$\mu-1.48\sigma=35$ (i)

And

$$P(X<63)=89\%=(\frac{89}{100})=0.89$$

The 89% items are under 63, the area to the left of the ordinate "Mean and X=63" is (0.89-0.5=0.39). Then Z_2 corresponding to the area 0.39 is 1.23

Using area table inversely

$$P(Z_2/P=0.39)=\frac{63-\mu}{\sigma}$$

$1.23=\frac{63-\mu}{\sigma}$ Sign is positive because right sided

$$1.23\sigma=63-\mu$$

$\mu+1.23\sigma=63$ (ii)

Subtracting equation (i) and (ii)

$$\mu-1.48\sigma=35$$
$$\underline{\mu+1.23\sigma=63}$$
$$-2.71\sigma=-28$$
$$\sigma=\frac{-28}{-2.71}=10.332$$

The value of standard deviation put in equation (ii) then we get the value of mean

$$\mu+1.23(10.3321)=63$$
$$\mu+12.7085=63$$
$$\mu=63-12.7085=50.2915$$

Example: In a normal distribution the first moment about 10 is 40 and the fourth moment about 50 is 48. What is the arithmetic mean, variance and standard deviation.
Solution: The relationship between first raw moment and mean is $Mean=A+\mu_1'$

Given that $\mu_1'=E(X-10)=40$ $\mu_4'=E(X-50)^4=48$

$D=(X-10)$ $A=10$

So $Mean=A+\mu_1'=10+40=50$

$\mu_4'=E(X-50)^4=48$ (i)

As we know that fourth moment about mean

$\mu_4=E(X-\mu)^4$ Therefore $Mean=\mu=50$, so it becomes

$\mu_4=E(X-50)^4$ (ii)

Comparing (i) and (ii)

$\mu_4' = \mu_4$

So, we get

$\mu_4 = 48$

As we know that

$\mu_4 = 3\sigma^4$

$48 = 3\sigma^4$

$\sigma^4 = \frac{48}{3} = 16$

$\sigma^4 = (2)^4$

Then

$\sigma = S\tan dard\ deviation = 2$

$\sigma^2 = Variance = 4$

Example: A manager leaves his house at 7 A.M for his office. The time taken to reach the office is normally distributed with mean 45 minutes and standard deviation of 10 minutes. What is the probability that he will be late for office on a randomly selected day if the office starts at 8 A.M?

Solution: Given that $\mu = 45$ and $\sigma = 10$

$P(X \geq 60 \min utes) = ?$

By S.N.V

$$Z = \frac{X-\mu}{\sigma} = \frac{60-45}{10} = \frac{15}{10} = 1.50$$

$\text{P(X} \geq 60) = \text{P(Z} \geq 1.50) = P(0 \leq Z \leq \infty) - P(0 \leq Z \leq 1.50) = 0.5 - 0.43319 = 0.0668$

Example: If "X" is normally distributed with mean 20 minutes and variance 16. Let the function $Y = 20X - 50$ is also normally distributed. Determine the probabilities

i) $P(Y \geq 510)$ ii) $P(Y \geq 250)$

Solution: As $X \to N(20,16)$ $\mu_X = 20$ $\sigma^2{}_X = 16$ $\sigma_X = 4$ $Y = 20X - 50$

First we find mean and standard deviation for "Y" distribution.

$E(Y) = E(20X - 50) = 20E(X) - 50 = 20(20) - 50 = 350 = \mu_Y$

$V(Y) = V(20X - 50) = (20)^2 V(X) + 0 = 400(16) = 6400 = \sigma_Y^2$

$\sigma_Y = 80$

Now $Y \to N(350,6400)$

i) $P(Y \geq 510) = ?$

By S.N.V

$$Z = \frac{Y-\mu_Y}{\sigma_Y} = \frac{510-350}{80} = \frac{160}{80} = 2.0$$

$\text{P(Y} \geq 510) = \text{P(Z} > 2.0) = P(0 < Z < \infty) - P(0 < Z < 2.0) = 0.5 - 0.47725 = 0.0228$

ii) $P(Y \geq 250) = ?$

By S.N.V

$$Z = \frac{Y-\mu_Y}{\sigma_Y} = \frac{250-350}{80} = \frac{-100}{80} = -1.25$$

$\text{P(Y} \geq 250) = \text{P(Z} > -1.25) = P(0 < Z < \infty) + P(-1.25 < Z < 0) = 0.5 + 0.39435 = 0.89435$

Example: The 10th percentiles and 90th percentiles of a certain normal distribution are 17.2 and 42.8 respectively. Find μ , σ , Q_1 , Q_3 and P_{50} .

Solution: Given that $P_{10} = 17.2$ and $P_{90} = 42.8$

i) $\mu = \frac{P_{10} + P_{90}}{2} = \frac{17.2 + 42.8}{2} = \frac{60}{2} = 30$ Because we know that mean lies exactly in the centre of P_{10} and P_{90} .

ii) $\sigma = ?$

P_{10} be a point contain ($10\% = (\frac{10}{100}) = 0.10$) values below the point. Then the area between the point P_{10} and mean=24 is (0.5-0.10=0.40

$$Z = \frac{P_{10} - 30}{\sigma}$$

$P(Z / P = 0.40) = \frac{1 - 30}{\sigma}$ Using area table inversely

$-1.28\sigma = -12.8$ Sign is negative because left sided

$\sigma = \frac{-12.8}{-1.28} = 10.0$

iii) $Q_1 = ?$

As we know that

$Q_1 = \mu - 0.6745\sigma$

$Q_1 = 30 - 0.6745(10) = 23.255$

iv) $Q_3 = ?$

$Q_3 = \mu + 0.6745\sigma$

$Q_3 = 30 + 0.6745(10) = 36.745$

Example: Find the probability that 500 tosses of a fair coin will result in

i) 240 or more ii) more than 240 heads iii) 240 or less heads
iv) Less than 240 heads v) Between 240 and 260 heads inclusive
vi) Between 240 and 260 heads vii) Exactly 240 heads

Solution: Given that $n = 500$ $P = 0.5$ $q = 0.5$

W know that in binomial distribution if $n \geq 30$ and $P \cong q$ then binomial distribution approaches to normal distribution.

$\mu = np = 500(0.5) = 250$

$\sigma = \sqrt{npq} = 5\sqrt{00(0.5)(0.5)} = 11..1803$

i) $P(X \geq 240) = P(X \geq 239.5) = ?$

By S.N.V

$Z = \frac{X - \mu}{\sigma} = \frac{239.5 - 250}{11.1803} = -0.94$

$\text{P(X} \geq 240) = \text{P(Z} \geq \text{-0.94)} = P(0 \leq Z \leq \infty) + P(-0.94 \leq Z \leq 0) = 0.5 + 0.32639 = 0.8264$

ii) $P(X > 240) = P(X > 240.5) = ?$

By S.N.V

$Z = \frac{X - \mu}{\sigma} = \frac{240.5 - 250}{11.1803} = -0.85$

$\text{P(X} > 240) = \text{P(Z} > \text{-0.85)} = P(0 < Z < \infty) + P(-0.85 < Z < 0) = 0.5 + 0.30234 = 0.80234$

iii) 240 or less heads

$P(X \leq 240) = P(X \leq 240.5) = ?$

By S.N.V

$Z = \frac{X - \mu}{\sigma} = \frac{240.5 - 250}{11.1803} = -0.85$

$\text{P(X} \leq 240) = \text{P(Z} \leq \text{-0.85)} = P(-\infty \leq Z \leq 0) - P(-0.85 \leq Z \leq 0) = 0.5 - 0.30234 = 0.1977$

iv) Less than 240 heads

$P(X < 240) = P(X < 239.5) = ?$

By S.N.V

$$Z = \frac{X-\mu}{\sigma} = \frac{239.5-250}{11.1803} = -0.94$$

P(X < 240) = P(Z < -0.94) $= P(-\infty < Z < 0) + P(-0.94 < Z < 0) = 0.5 - 0.32639 = 0.1736$

v) Between 240 and 260 inclusive

$P(240 \le X \le 260) = P(239.5 \le X \le 260.5) = ?$

By S.N.V

$$Z_1 = \frac{X_1-\mu}{\sigma} = \frac{239.5-250}{11.1803} = -0.94$$

$$Z_2 = \frac{X_2-\mu}{\sigma} = \frac{260.5-250}{11.1803} = 0.94$$

$$P(240 \le X \le 260) = P(-0.94 \le Z \le 0.94) = P(-0.94 \le Z \le 0) + P(0 \le Z \le 0.94)$$
$$= 0.32639 + 0.32639 = 0.6528$$

vi) Between 240 and 260 heads

$P(240 < X < 260) = P(240.5 < X < 259.5) = ?$

By S.N.V

$$Z_1 = \frac{X_1-\mu}{\sigma} = \frac{240.5-250}{11.1803} = -0.85$$

$$Z_2 = \frac{X_2-\mu}{\sigma} = \frac{259.5-250}{11.1803} = 0.85$$

$$P(240 < X < 260) = P(-0.85 \le Z \le 0.85) = P(-0.85 \le Z \le 0) + P(0 \le Z \le 0.85)$$
$$= 0.30234 + 0.30234 = 0.6047$$

vii) Exactly 240 heads

$P(X = 240) = P(239.5 \le X \le 240.5) = ?$

By S.N.V

$$Z_1 = \frac{X_1-\mu}{\sigma} = \frac{239.5-250}{11.1803} = -0.94$$

$$Z_2 = \frac{X_2-\mu}{\sigma} = \frac{240.5-250}{11.1803} = -0.85$$

$$P(239.5 \le X \le 240.5) = P(-0.94 \le Z \le -0.85) = P(-0.94 \le Z \le 0) - P(-0.85 \le Z \le 0)$$
$$= 0.32639 - 0.30234 = 0.0241$$

Q.9.15 (a): if $f(x) = ke^{-(x^2-6x+9)/24}$ is the equation of a normal curve find the value of "k" and mean and standard deviation.

Solution: give that $f(x) = ke^{-(x^2-6x+9)/24}$

$$f(x) = ke^{-(x-3)^2/2(12)} \qquad \text{(A)}$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2(\sigma^2)} \qquad \text{(B)}$$

Comparing (A) and (B)

$$K = \frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{\sqrt{24\pi}} \quad \sigma^2 = 12 \qquad \sigma = \sqrt{12} \qquad \mu = 3$$

Q.16: Prove that for the normal distribution, the quartile deviation, the mean deviation and standard deviation are approximately in the ratio $10:12:15$

Proof: As we know that

$$Q.D = \frac{2}{3}\sigma \qquad M.D = \frac{4}{5}\sigma$$

$$3Q.D = 2\sigma$$

Dividing by 30 in each side

$$\frac{3}{30}Q.D = \frac{2}{30}\sigma$$

$$\frac{1}{10}Q.D = \frac{1}{15}\sigma \qquad \text{(A)}$$

$$5M.D = 4\sigma$$

Dividing by 60 in each side

$$\frac{5}{60}M.D = \frac{4}{60}\sigma$$

$$\frac{1}{12}M.D = \frac{1}{15}\sigma \qquad \text{(B)}$$

From (A) and (B)

$\frac{1}{10}Q.D = \frac{1}{12}M.D = \frac{1}{15}\sigma$ Hence the Q.D, M.D and S.D are approximately in the ratio 10:12:15 Proved

Q.9.24 (b): If the m.g.f of "X" is $M(t) = e^{-6t+32t^2}$ then find $P(-4 \le X < 16)$ *and* $P(-10 < X < 0)$

Solution: Given that $M(t) = e^{-6t+32t^2}$

$$M(t) = e^{-6t+\frac{64}{2}t^2} \qquad \text{(A)}$$

$$M(t) = e^{\mu t+\frac{1}{2}\sigma^2 t^2} \qquad \text{(B)}$$

Comparing (A) and (B) and we get

$\mu = -6$ *and* $\sigma^2 = 64$

Remaining parts do yourself

Fitting a normal distribution

Method-I

If the observed frequency distribution is given then find mean and standard deviation

i) $\overline{X} = \frac{\sum fx}{\sum f}$ *And* $S = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$

ii) Find "Z" values for all upper class boundaries

iii) Find cumulative probabilities $\phi(Z)$

iv) Obtain $\hat{P}$ for each class subtraction

v) Expected or theoretical frequencies $= N\hat{P}$

This procedure is called fitting normal distribution by area table

Method-II

If only μ *and* σ^2 is given

i) First we find the range of $\mu \pm 3\,\sigma$

ii) Rake appropriate "class interval=h" for making classes between 6 and 15

iii) Find "Z" values for all upper class boundaries

iv) Find cumulative probabilities $\phi(Z)$

v) Obtain $\hat{P}$ for each class subtraction

vi) Expected or theoretical frequencies $= N\hat{P}$

Q.42 (b): Fit a normal distribution given mean=27.0 standard deviation is 2.2 and total frequency is 209.

Solution: Given that $\mu = 27.0$ $\sigma = 2.2$ $N = 209$

Since the bulk (mean maximum values lies between the interval) of normal distribution $\mu \pm 3\,\sigma$ then the interval $27 \pm 3(2.2)$ is $(20.4, 33.6)$

$Range = 33.6 - 20.4 = 13.2$

Classes should be between 6 and 15 so we assume 7 classes

$$h = Class\ \text{int}\,erval = \frac{Range}{no.of\ classes} = \frac{13.2}{7} = 1.885 = 2.0$$

Then we make the class

Classes	UCB	$Z = \frac{UCB - \mu}{\sigma}$	$\phi(Z)$	$\hat{P}$	Expected Frequency $= N\hat{P}$
20-22	22	-2.27	0.0116	0.0116	2.4244
22-24	24	-1.36	0.0869	$0.0869 - 0.0116 = 0.0753$	15.7377
24-26	26	-0.45	0.3264	$0.3264 - 0.0869 = 0.2395$	50.0555
26-28	28	0.45	0.6736	0.3472	72.5648
28-30	30	1.36	0.9131	0.2395	50.0555
30-32	32	2.27	0.9884	0.0753	15.0555
32-34	∞	∞	1.000	0.0116	2.4244
				1.0	209

Q.9.43: The following table gives the distribution of statures among the first year students of a university

Stature(X)	f	fx	fx^2
61	2	122	7442
62	10	620	38440
63	11	693	43659
64	38	2432	155648
65	57	3705	240825
66	93	6138	405108
67	106	7102	475834
68	126	8568	582624
69	109	7521	518949
70	87	6090	426300
71	75	5325	378075
72	23	1656	119232
73	9	657	47961
74	4	296	21904
	750	50925	3462001

a) Test the normality of the distribution by comparing the proportion of the cases lying between $\bar{x} \pm S$, $\bar{x} \pm 2S$, $\bar{x} \pm 3S$ for distribution and for the normal distribution

$$\bar{X} = \frac{\sum fx}{\sum f} = 67.9 \qquad S.D = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = 2.365$$

Number of stature between $\bar{x} \pm S$ the interval (65.54, 70.26)

Stature(X)	f
66	93
67	106
68	126
69	109
70	87
	521

The percentage no. of observation falling between the intervals

$$\bar{x} \pm S = \frac{521}{750} \times 100 = 69.5\ \%$$

Number of stature between $\bar{x} \pm 2S$ the interval (62.77, 72.23)

Stature(X)	f
63	11
64	38
65	57
66	93
67	106
68	126
69	109
70	87
71	75
72	23
	725

The percentage no. of observation falling between the intervals

$$\bar{x} \pm 2S = \frac{725}{750} \times 100 = 96.67\%$$

Number of stature between $\bar{x} \pm 3S$ the interval (60.405, 74.595)

Stature(X)	f
61	2
62	10
63	11
64	38
65	57
66	93
67	106
68	126
69	109
70	87
71	75
72	23
73	9
74	4
	750

The percentage no. of observation falling between the intervals

$$\bar{x} \pm 3S = \frac{750}{750} \times 100 = 100\ \%$$

b)

Class boundary	Ucb	$Z=\frac{Ucb-\overline{X}}{S}$	$\phi(Z)$	$\hat{P}$	Expected Frequency $=N\hat{P}$
60.5-61.5	61.5	-2.71	0.0034	0.0034	2.55
61.5-62.5	62.5	-2.28	0.0113	$0.0113-0.0034=0.0079$	5.925
62.5-63.5	63.5	-1.86	0.0314	$0.0314-0.0113=0.0201$	15.075
63.5-64.5	64.5	-1.44	0.0749	0.0435	32.625
64.5-65.5	65.5	-1.01	0.1562	0.0813	60.975
65.5-66.5	66.5	-0.59	0.2776	0.1214	91.05
66.5-67.5	67.5	-0.17	0.4325	0.1549	116.175
67.5-68.5	68.5	0.25	0.5987	0.1662	124.65
68.5-69.5	69.5	0.68	0.7518	0.1531	114.825
69.5-70.5	70.5	1.10	0.8643	0.1125	84.375
70.5-71.5	71.5	1.52	0.9357	0.0714	53.55
71.5-72.5	72.5	1.94	0.9738	0.0381	28.575
72.5-73.5	73.5	2.37	0.9911	0.0173	12.975
73.5-74.5	∞	∞	1.000	0.0089	6.675
				1.0	750

Example 9.23: Find the ordinate of the frequency distribution of weights

Weight(Kg)	f	X	fx	fx^2
28-31	1	29.5		
32-35	14	33.5		
36-39	56	37.5		
40-43	172	41.5		
44-47	245	45.5		
47-51	263	49.5		
52-55	156	53.5		
56-59	67	57.5		
60-63	23	61.5		
64-67	3	65.5		

Solution: First we calculate mean and standard deviation of the given frequency distribution

$$\overline{X}=\frac{\sum fx}{\sum f}=47.71 \qquad S.D=\sqrt{\frac{\sum fx^2}{\sum f}-\left(\frac{\sum fx}{\sum f}\right)^2}=5.88$$

The procedure to find the ordinates are given below

X	$Z=\frac{X-\overline{X}}{s}$	$\phi(Z)$	$Ordinates=\frac{nh}{s}\phi(z)=\frac{1000(4)}{5.88}\phi(z)$
29.5	-3.10	0.0033	2.245
33.5	-2.42	0.0213	14.49
37.5	-1.74	0.0878	59.73
41.5	-1.06	0.2275	154.76
45.5	-0.38	0.3712	252.52
49.5	0.30	0.3814	259.46
53.5	0.98	0.2468	167.89
57.5	1.66	0.1006	68.44
61.5	2.35	0.0252	17.14
65.5	3.03	0.0041	2.79

Short Question

Q.1: Give the background of normal distribution?

Ans: Normal distribution is also called "Gaussian distribution in the honor of Karl F Gauss who derived its equation. Karl Pearson named it normal distribution, in 1893. It was discovered by "Abraham de moivre (1667-1754) as a limiting form of binomial distribution, when $n \geq 30$ and $p \equiv q$

Q.2: Define normal distribution?

Ans: Normal distribution is defined as limiting form of binomial distribution when $n \geq 30$ and $p \equiv q$. It is distribution of continuous random variable. Its p.d.f (Probability density function) is given as $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \qquad -\infty \leq X \leq +\infty$

Where $\mu = Mean \qquad \sigma = S\tan dard\ deviation$

$\pi = Cons\tan t\ Approximately\ equal to = \frac{22}{7} = 3.14159$

$e = Cons\tan t\ Approximately\ equal\ to = 2.71828$

$X = Abscissa\ \ i.e. value\ marked\ on\ X - axis$

$Y = Ordinate\ hieght\ \ i.e. value\ marked\ on\ Y - axis$

Or

Define normal distribution

Ans: Let "X" be a continuous random variable with interval $(-\infty, +\infty)$ is said to be normal distribution having its probability density function (p.d.f) is given as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \qquad -\infty \leq X \leq +\infty$$

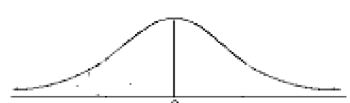
It has two parameters (μ, σ^2).

Q.3: What is the range of normal variable?

Ans: The range of normal variable "X" from $(-\infty\ to\ +\infty)$ i.e $-\infty \leq Z \leq +\infty$

Q.4: What is the shape of normal curve?

Ans: The shape of normal curve is uni-modal, symmetrical and bell shaped as shown in the figure.



Q.5: What is the role of mean" μ " and standard deviation " σ " in the normal curve?

Ans: Mean" μ " and variance " σ^2 "are the parameters of normal distribution. Where $\mu = Mean$ is known as shape parameters and σ is known as scale parameters.

Q.8: When normal distribution changes into standard normal distribution?

Ans: When the normal random variable "X" is expressed in terms of deviations from mean in units of standard deviation the normal distribution changes into standard normal distribution.

Q.9: How much area of the normal curve lies between mean+2Standard deviations $(\mu + 2\sigma)$?

Ans: We know that 95.44% area lies between $(\mu \pm 2\sigma)$. It means $(\mu + 2\sigma)$ contains $\left(\frac{95.44}{2}\right)\% = 47.72\%$ area.

Q.10: What is the relationship between mean deviation and standard deviation of normal curve?

Ans: The relationship between mean deviation and standard deviation is given by $M.D = 0.7979\sigma = \sigma\sqrt{\frac{2}{\pi}}$ in normal distribution.

Q.11: Write down the ordinates of the standard normal curve at a) Z=1 b) Z=-1

Ans: The ordinate of Z=1 is 0.3413 and Z=-1 is 0.3413

Q.12: Define points of inflection in a normal distribution.

Ans: Those points at which concavity of curve changes are called points of inflection. There are two points of inflection which are equidistant from mean. The two points ate given as $\left(\mu-\sigma,\frac{1}{\sigma\sqrt{2\pi e}}\right)$ and $\left(\mu+\sigma,\frac{1}{\sigma\sqrt{2\pi e}}\right)$

Q.13 What is the relationship between the binomial distribution and the normal distribution?

Ans: In case of binomial distribution when $n \geq 30$ and $p \equiv q$ then it becomes normal distribution. It is denoted by $b(x;n,p) \to N(\mu,\sigma^2)$ when $n \geq 30$ and $p \equiv q$

Q.14: Explain why odd order moments about mean equal to zero for the normal distribution.

Ans: In normal curve, if nth moment is odd, the value of odd moment will always equal to zero. This is because the normal curve is symmetrical and for symmetrical distribution sum of the positive deviations from μ will always equal to sum of the negative deviations from μ and these deviations are cancel out to each other.

Q.15: When it is appropriate to use a normal approximation to the binomial distribution?

Ans: In case of binomial distribution when $n \geq 30$ and $p \equiv q$ then it is appropriate to use normal distribution.

Q.16: Records from a dental practice show that the probability of waiting to go into the surgery for more than 20 minutes is 0.0239. If the waiting time is normally distributed with standard deviation 3.78 minutes, find the mean waiting time.

Solution: In normal distribution it is given that

$P(X > 20) = 0.0239$

By S.N.V

$Z = \frac{X-\mu}{\sigma}$

$P(Z/P = 0.4761) = \frac{20-\mu}{3.78}$ Using area table inversely

$1.98 = \frac{20-\mu}{3.78}$ Sign is positive because right sided

$3.78(1.98) = 20-\mu$

$\mu = 20 - 7.4844 = 12.5156$

Q.17: A manufacturer of pipe knows that the pipe lengths it produces vary diameter and that the diameters are normally distributed. The mean diameter is "1" inch and the probability that a length of pipe will have a diameter exceeding 1.1 inches is 0.1587. Find the variance of the diameters.

Solution: In normal distribution it is given that

$P(X > 20) = 0.0239$

By S.N.V

$Z = \frac{X-\mu}{\sigma}$

$P(Z/P = 0.3413) = \frac{1.1-1}{\sigma}$ Using area table inversely

$1.0 = \frac{1.1-1}{\sigma} = \frac{0.1}{\sigma}$ Sign is positive because right sided

$\sigma = 0.1$ And variance is $\sigma^2 = 0.010$