Math-21201: Advanced Real Analysis

Prerequisites: Real Analysis

Specific Objectives of Course: A continuation of real analysis, this course will continue to cover the fundamentals of real analysis, concentrating on the Riemann-Stieltjes integrals, functions of bounded variation, improper integrals and convergence of series. Emphasis would be on proofs of main results.

Course Outlines:

The Riemann-Stieltjes integrals: Existence of integrals, properties of integrals, fundamental theorem of calculus and its applications, change of variable theorem, integration by parts.

Functions of bounded variation: Properties of functions of bounded variation.

Improper integrals: Types of improper integrals, tests for convergence of improper integrals, beta and gamma functions, absolute and conditional convergence of improper integrals.

Sequences and series of functions: Power series, point-wise and uniform convergence, uniform convergence and continuity, uniform convergence and differentiation, examples of uniform convergence.

Recommended Books:

1) W. Rudin, Principles of Mathematical Analysis, 3rd Ed., McGraw-Hill (2012)
2) K. R. Davidson and A. P. Donsig, Real Analysis with Real Applications, Prentice Hall Inc., Upper Saddle River (2011)
3) G. B. Folland, Real Analysis, 2nd Edition, John Wiley and Sons, New York (2012)
4) E. Hewitt and K. Stromberg, Real and Abstract Analysis, Springer-Verlag, Berlin Heidelberg, New York (2011)
5) H. L. Royden, Real Analysis, 3rd Edition, Mac Millan, New York (2010)

Chapter 6 – Riemann-Stieltjes Integral.

➢ Introduction

In elementary treatment of Integral Calculus the subject of integration is treated as inverse of differentiation. The subject arose in connection with the determination of areas of plane regions and was based on the notion of the limit of a type of sum when the number of terms in the sum tends to infinity and each term tends to zero. In fact the name Integral Calculus has its origin in this process of summation. It was only afterwards that it was seen that the subject of integration can also be viewed from the point of the inverse of differentiation.

➢ Partition

Let $[a,b]$ be a given interval. A finite set $P=\{a=x_0,x_1,x_2,....,x_k,....,x_n=b\}$ is said to be a partition of $[a,b]$ which divides it into n such intervals

$$[x_0,x_1],[x_1,x_2],[x_2,x_3],......,[x_{n-1},x_n]$$

Each sub-interval is called a *component* of the partition.

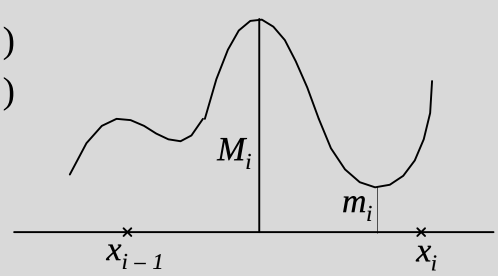
Obviously, corresponding to different choices of the points x_i we shall have different partition.

The maximum of the length of the components is defined as the *norm* of the partition.

➢ Riemann Integral

Let f be a real-valued function defined and bounded on $[a,b]$. Corresponding to each partition P of $[a,b]$, we put

$$M_i=\sup f(x) \qquad (x_{i-1}\le x\le x_i)$$
$$m_i=\inf f(x) \qquad (x_{i-1}\le x\le x_i)$$



We define upper and lower sums as

$$U(P,f)=\sum_{i=1}^{n}M_i\,\Delta x_i$$

and $$L(P,f)=\sum_{i=1}^{n}m_i\,\Delta x_i$$

where $\Delta x_i=x_i-x_{i-1} \quad (i=1,2,....,n)$

and finally $$\overline{\int_a^b} f\,dx=\inf U(P,f) \;\ldots\ldots\ldots\ldots\ldots\; (i)$$

$$\underline{\int_a^b} f\,dx=\sup L(P,f) \;\ldots\ldots\ldots\ldots\ldots\; (ii)$$

Where the infimum and the supremum are taken over all partitions P of $[a,b]$.

Then $\overline{\int_a^b} f\,dx$ and $\underline{\int_a^b} f\,dx$ are called the upper and lower Riemann Integrals of f over $[a,b]$ respectively.

In case the upper and lower integrals are equal, we say that f is Riemann-Integrable on $[a,b]$ and we write $f\in \boldsymbol{R}$, where $\boldsymbol{R}$ denotes the set of Riemann integrable functions.

The common value of (i) and (ii) is denoted by $\int_a^b f\,dx$ or by $\int_a^b f(x)\,dx$.

Which is known as the Riemann integral of f over $[a,b]$.

➢ Theorem

The upper and lower integrals are defined for every bounded function f.

Proof

Take M and m to be the upper and lower bounds of $f(x)$ in $[a,b]$.

$$\Rightarrow \quad m \le f(x) \le M \qquad (a \le x \le b)$$

Then $M_i \le M$ and $m_i \ge m$ $\qquad (i = 1, 2, \ldots\ldots, n)$

Where M_i and m_i denote the supremum and infimum of $f(x)$ in (x_{i-1}, x_i) for certain partition P of $[a,b]$.

$$\Rightarrow \quad L(P,f) = \sum_{i=1}^{n} m_i\,\Delta x_i \ge \sum_{i=1}^{n} m\,\Delta x_i \qquad (\Delta x_i = x_{i-1} - x_i)$$

$$\Rightarrow \quad L(P,f) \ge m\sum_{i=1}^{n}\Delta x_i$$

But $\displaystyle\sum_{i=1}^{n}\Delta x_i = (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + \ldots. + (x_n - x_{n-1})$

$$= x_n - x_0 = b - a$$

$$\Rightarrow \quad L(P,f) \ge m(b-a)$$

Similarity $U(P,f) \le M(b-a)$

$$\Rightarrow \quad m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

Which shows that the numbers $L(P,f)$ and $U(P,f)$ form a bounded set.

$\Rightarrow$ The upper and lower integrals are defined for every bounded function f. ⊙

➢ Riemann-Stieltjes Integral

It is a generalization of the Riemann Integral. Let $\alpha(x)$ be a monotonically increasing function on $[a,b]$. $\alpha(a)$ and $\alpha(b)$ being finite, it follows that $\alpha(x)$ is bounded on $[a,b]$. Corresponding to each partition P of $[a,b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

(Difference of values of α at x_i & x_{i-1})

$\because$ $\alpha(x)$ is monotonically increasing.

$\therefore$ $\Delta\alpha_i \ge 0$

Let f be a real function which is bounded on $[a,b]$.

Put $\qquad U(P,f,\alpha) = \displaystyle\sum_{i=1}^{n} M_i\,\Delta\alpha_i$

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i\,\Delta\alpha_i$$

Where M_i and m_i have their usual meanings.

Define

$$\int_a^{\bar{b}} f\,d\alpha = \inf U(P,f,\alpha) \;\ldots\ldots\ldots\ldots\ldots\; (i)$$

$$\int_{\underline{a}}^{b} f\,d\alpha = \sup L(P,f,\alpha) \;\ldots\ldots\ldots\ldots\ldots\; (ii)$$

Where the infimum and supremum are taken over all partitions of $[a,b]$.

If $\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$, we denote their common value by $\int_a^b f\,d\alpha$ or $\int_a^b f(x)\,d\alpha(x)$.

This is the Riemann-Stieltjes integral or simply the Stieltjes Integral of f w.r.t. α over $[a,b]$.

If $\int_a^b f\,d\alpha$ exists, we say that f is integrable w.r.t. α, in the Riemann sense, and write $f \in \boldsymbol{R}(\alpha)$.

➢ *Note*

The Riemann-integral is a special case of the Riemann-Stieltjes integral when we take $\alpha(x) = x$.

$\because$ The integral depends upon f, α, a and b but not on the variable of integration.

$\therefore$ We can omit the variable and prefer to write $\int_a^b f\,d\alpha$ instead of $\int_a^b f(x)\,d\alpha(x)$.

In the following discussion f will be assume to be real and bounded, and α monotonically increasing on $[a,b]$.

➢ *Refinement of a Partition*

Let P and P^* be two partitions of an interval $[a,b]$ such that $P \subset P^*$ i.e. every point of P is a point of P^*, then P^* is said to be a *refinement* of P.

➢ *Common Refinement*

Let P_1 and P_2 be two partitions of $[a,b]$. Then a partition P^* is said to be their *common refinement* if $P^* = P_1 \cup P_2$.

➢ *Theorem*

If P^* is a refinement of P, then

$$L(P, f, \alpha) \le L(P^*, f, \alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$$\text{and} \quad U(P, f, \alpha) \ge U(P^*, f, \alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

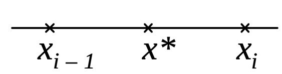
Proof

Let us suppose that P^* contains just one point x^* more than P such that $x_{i-1} < x^* < x_i$ where x_{i-1} and x_i are two consecutive points of P.

Put

$$w_1 = \inf f(x) \qquad (x_{i-1} \le x \le x^*)$$

$$w_2 = \inf f(x) \qquad (x^* \le x \le x_i)$$



It is clear that $w_1 \ge m_i$ & $w_2 \ge m_i$ where $m_i = \inf f(x)$, $(x_{i-1} \le x \le x_i)$.

Hence

$$\begin{aligned} L(P^*, f, \alpha) - L(P, f, \alpha) &= w_1\left[\alpha(x^*) - \alpha(x_{i-1})\right] + w_2\left[\alpha(x_i) - \alpha(x^*)\right] \\ &\quad - m_i\left[\alpha(x_i) - \alpha(x_{i-1})\right] \\ &= w_1\left[\alpha(x^*) - \alpha(x_{i-1})\right] + w_2\left[\alpha(x_i) - \alpha(x^*)\right] \\ &\quad - m_i\left[\alpha(x_i) - \alpha(x^*) + \alpha(x^*) - \alpha(x_{i-1})\right] \\ &= (w_1 - m_i)\left[\alpha(x^*) - \alpha(x_{i-1})\right] + (w_2 - m_i)\left[\alpha(x_i) - \alpha(x^*)\right] \end{aligned}$$

$\because$ α is a monotonically increasing function.

$\therefore$ $\alpha(x^*)-\alpha(x_{i-1})\geq 0$, $\alpha(x_i)-\alpha(x^*)\geq 0$

$$\Rightarrow L(P^*,f,\alpha)-L(P,f,\alpha)\ \geq\ 0$$

$$\Rightarrow L(P,f,\alpha)\ \leq\ L(P^*,f,\alpha) \qquad \text{which is } (i)$$

If P^* contains k points more than P, we repeat this reasoning k times and arrive at (i).

Now put

$$W_1=\sup f(x) \qquad (x_{i-1}\leq x\leq x^*)$$

and $$W_2=\sup f(x) \qquad (x^*\leq x\leq x_i)$$

Clearly $M_i\geq W_1$ & $M_i\geq W_2$

Consider

$$\begin{aligned}U(P,f,\alpha)-U(P^*,f,\alpha)&=M_i[\alpha(x_i)-\alpha(x_{i-1})]\\&\quad-W_1[\alpha(x^*)-\alpha(x_{i-1})]-W_2[\alpha(x_i)-\alpha(x^*)]\\&=M_i[\alpha(x_i)-\alpha(x^*)+\alpha(x^*)-\alpha(x_{i-1})]\\&\quad-W_1[\alpha(x^*)-\alpha(x_{i-1})]-W_2[\alpha(x_i)-\alpha(x^*)]\\&=(M_i-W_1)[\alpha(x^*)-\alpha(x_{i-1})]+(M_i-W_2)[\alpha(x_i)-\alpha(x^*)]\ \geq\ 0\\&\qquad(\because\ \alpha \text{ is}\uparrow)\end{aligned}$$

$$\Rightarrow U(P,f,\alpha)\ \geq\ U(P^*,f,\alpha) \qquad \text{which is } (ii)$$

⊙

➢ Theorem

Let f be a real valued function defined on $[a,b]$ and α be a monotonically increasing function on $[a,b]$. Then

$$\sup L(P,f,\alpha)\ \leq\ \inf U(P,f,\alpha)$$

$$\text{i.e.}\quad \underline{\int_a^b} f\,d\alpha\ \leq\ \overline{\int_a^b} f\,d\alpha$$

Proof

Let P^* be the common refinement of two partitions P_1 and P_2. Then

$$L(P_1,f,\alpha)\ \leq\ L(P^*,f,\alpha)\ \leq\ U(P^*,f,\alpha)\ \leq\ U(P_2,f,\alpha)$$

Hence $L(P_1,f,\alpha)\ \leq\ U(P_2,f,\alpha)$ …………. (i)

If P_2 is fixed and the supremum is taken over all P_1 then (i) gives

$$\underline{\int_a^b} f\,d\alpha\ \leq\ U(P_2,f,\alpha)$$

Now take the infimum over all P_2

$$\Rightarrow \underline{\int_a^b} f\,d\alpha\ \leq\ \overline{\int_a^b} f\,d\alpha$$

⊙

❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀

> **Theorem (Condition of Integrability or Cauchy's Criterion for Integrability.)**
> $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$ iff for every $\varepsilon > 0$ there exists a partition P such that
> $$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

Proof

Let $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ .... (i)

Then $L(P,f,\alpha) \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha \le U(P,f,\alpha)$

$$\Rightarrow \underline{\int_a^b} f\,d\alpha - L(P,f,\alpha) \ge 0 \quad \text{and} \quad U(P,f,\alpha) - \overline{\int_a^b} f\,d\alpha \ge 0$$

Adding these two results, we have

$$\underline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f\,d\alpha - L(P,f,\alpha) + U(P,f,\alpha) \ge 0$$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \le U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \qquad \text{from } (i)$$

i.e. $0 \le \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \varepsilon$ for every $\varepsilon > 0$.

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha \qquad \text{i.e.} \quad f \in \boldsymbol{R}(\alpha)$$

Conversely, let $f \in \boldsymbol{R}(\alpha)$ and let $\varepsilon > 0$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha$$

Now $\overline{\int_a^b} f\,d\alpha = \inf U(P,f,\alpha)$ and $\underline{\int_a^b} f\,d\alpha = \sup L(P,f,\alpha)$

There exist partitions P_1 and P_2 such that

$$U(P_2,f,\alpha) - \int_a^b f\,d\alpha < \frac{\varepsilon}{2} \quad \text{.............. } (ii)$$

and $$\int_a^b f\,d\alpha - L(P_1,f,\alpha) < \frac{\varepsilon}{2} \quad \text{.............. } (iii)$$

$U(P_2,f,\alpha) - \varepsilon/2 < \int f\,d\alpha$

$\int f\,d\alpha < L(P_1,f,\alpha) + \varepsilon/2$

We choose P to be the common refinement of P_1 and P_2.

Then

$$U(P,f,\alpha) \le U(P_2,f,\alpha) < \int_a^b f\,d\alpha + \frac{\varepsilon}{2} < L(P_1,f,\alpha) + \varepsilon \le L(P,f,\alpha) + \varepsilon$$

So that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$ ⊙

➢ ***Theorem***

a) If $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$ holds for some P and some ε, then it holds (with the same ε) for every refinement of P.

b) If $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$ holds for $P=\{x_0,\ldots,x_n\}$ and s_i, t_i are arbitrary points in $[x_{i-1},x_i]$, then

$$\sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i<\varepsilon$$

c) If $f\in \boldsymbol{R}(\alpha)$ and the hypotheses of (*b*) holds, then

$$\left|\sum_{i=1}^{n}f(t_i)\Delta\alpha_i-\int_a^b f\,d\alpha\right|<\varepsilon$$

Proof

a) Let P^* be a refinement of P. Then

$$L(P,f,\alpha)\ \le\ L(P^*,f,\alpha)$$

$$\text{and}\quad U(P^*,f,\alpha)\ \le\ U(P,f,\alpha)$$

$$\Rightarrow\ L(P,f,\alpha)+U(P^*,f,\alpha)\ \le\ L(P^*,f,\alpha)+U(P,f,\alpha)$$

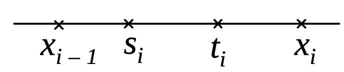
$$\Rightarrow\ U(P^*,f,\alpha)-L(P^*,f,\alpha)\ \le\ U(P,f,\alpha)-L(P,f,\alpha)$$

$$\because\ U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$$

$$\therefore\ U(P^*,f,\alpha)-L(P^*,f,\alpha)<\varepsilon$$

b) $P=\{x_0,\ldots,x_n\}$ and s_i, t_i are arbitrary points in $[x_{i-1},x_i]$.

$$\Rightarrow\ f(s_i)\text{ and } f(t_i)\text{ both lie in }[m_i,M_i].$$



$$\Rightarrow\ |f(s_i)-f(t_i)|\ \le\ M_i-m_i$$

$$\Rightarrow\ |f(s_i)-f(t_i)|\Delta\alpha_i\ \le\ M_i\Delta\alpha_i-m_i\Delta\alpha_i$$

$$\Rightarrow\ \sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i\ \le\ \sum_{i=1}^{n}M_i\,\Delta\alpha_i-\sum_{i=1}^{n}m_i\,\Delta\alpha_i$$

$$\Rightarrow\ \sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i\ \le\ U(P,f,\alpha)-L(P,f,\alpha)$$

$$\because\ U(P,f,\alpha)-L(P,f,\alpha)\ <\ \varepsilon$$

$$\therefore\ \sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i\ <\ \varepsilon$$

c) $\because\ m_i\ \le\ f(t_i)\ \le\ M_i$

$$\therefore\ \sum m_i\,\Delta\alpha_i\ \le\ \sum f(t_i)\Delta\alpha_i\ \le\ \sum M_i\,\Delta\alpha_i$$

$$\Rightarrow\ L(P,f,\alpha)\ \le\ \sum f(t_i)\Delta\alpha_i\ \le\ U(P,f,\alpha)$$

and also $L(P,f,\alpha)\ \le\ \int_a^b f\,d\alpha\ \le\ U(P,f,\alpha)$

Using (*b*), we have

$$\left|\sum f(t_i)\Delta\alpha_i-\int_a^b f\,d\alpha\right|\ <\ \varepsilon \qquad \odot$$

ꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷꟷ

> **Theorem**
> If f is continuous on $[a,b]$ then $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$.

Proof

Let $\varepsilon > 0$ be given. Choose $\beta > 0$ so that

$$[\alpha(b) - \alpha(a)]\beta < \varepsilon$$

f is continuous on $[a,b]$ $\Rightarrow$ f is uniformly continuous on $[a,b]$.

$\Rightarrow$ There exists a $\delta > 0$ such that

$$\left| f(s) - f(t) \right| < \beta \quad \text{if } x \in [a,b],\ t \in [a,b] \text{ and } \left| x - t \right| < \delta \quad \ldots\ldots\ldots(i)$$

If P is any partition of $[a,b]$ such that $\Delta x_i < \delta$ for all i

then (i) implies that $M_i - m_i \le \beta$, $(i = 1, 2, \ldots., n)$

$$\Rightarrow U(P, f, \alpha) - L(P, f, \alpha) = \sum M_i\, \Delta\alpha_i - \sum m_i\, \Delta\alpha_i$$

$$= \sum (M_i - m_i)\Delta\alpha_i$$

$$\le \beta \sum \Delta\alpha_i = \beta[\alpha(b) - \alpha(a)] < \varepsilon$$

$\Rightarrow$ $f \in \boldsymbol{R}(\alpha)$ by Cauchy Criterion. ⊙

> **Theorem**
> If f is monotonic on $[a,b]$, and if α is continuous on $[a,b]$, then $f \in \boldsymbol{R}(\alpha)$.
> (Monotonicity of α still assumed.)

Proof

Let $\varepsilon > 0$ be a given positive number.

For any positive integer n, choose a partition $P = \{x_0, x_1, \ldots.., x_n\}$ of $[a,b]$ such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n} \quad , \quad i = 1, 2, \ldots., n$$

This is possible because α is continuous and monotonic increasing on the closed interval $[a,b]$ and thus assumes every value between its bounds, $\alpha(a)$ and $\alpha(b)$.

Let f be monotonic increasing on $[a,b]$, so that its lower and upper bounds m_i, M_i in $[x_{i-1}, x_i]$ are given by

$$m_i = f(x_{i-1}) \quad , \quad M_i = f(x_i) \quad , \quad i = 1, 2, \ldots., n$$

$$\therefore\ U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$$= \frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)]$$

$$< \varepsilon \qquad \text{if } n \text{ is taken large enough.}$$

$\Rightarrow$ $f \in R(\alpha)$ on $[a,b]$.

⊙

Note: $f \in \boldsymbol{R}(\alpha)$ when either

i) f is continuous and α is monotonic, or

ii) f is monotonic and α is continuous, of course α is still monotonic.

➢ *Properties of Integral*

i) If $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$, then $cf \in \boldsymbol{R}(\alpha)$ for every constant c and

$$\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha .$$

Proof

$\because\ f \in \boldsymbol{R}(\alpha)$

$\therefore\ \exists$ a partition P such that

$$U(P, f,\alpha) - L(P, f,\alpha) < \varepsilon \quad , \text{ where } \varepsilon \text{ is an arbitrary +ive number.}$$

Now $\quad U(P,cf,\alpha) = \sum_{i=1}^{n} cM_i\,\Delta\alpha_i = c\sum_{i=1}^{n} M_i\,\Delta\alpha_i$

$\quad \& \quad L(P,cf,\alpha) = \sum_{i=1}^{n} cm_i\,\Delta\alpha_i = c\sum_{i=1}^{n} m_i\,\Delta\alpha_i$

$$\begin{aligned}\Rightarrow\ U(P, f,\alpha) - L(P, f,\alpha) &= c\left[\sum M_i\,\Delta\alpha_i - \sum m_i\,\Delta\alpha_i\right]\\ &= c\left[U(P, f,\alpha) - L(P, f,\alpha)\right]\\ &< c\varepsilon = \varepsilon_1\end{aligned}$$

$\Rightarrow\ cf \in \boldsymbol{R}(\alpha)$

$\because\ U(P,cf,\alpha) = c\left[U(P, f,\alpha)\right] \quad \& \quad L(P,cf,\alpha) = c\left[L(P, f,\alpha)\right]$

$\therefore\ \inf U(P,cf,\alpha) = c\left[\inf U(P, f,\alpha)\right] \ \&\ \sup L(P,cf,\alpha) = c\left[\sup L(P, f,\alpha)\right]$

where infimum and supremum are taken over all P on $[a,b]$.

$$\Rightarrow \overline{\int_a^b} cf\, d\alpha = c\overline{\int_a^b} f\, d\alpha \quad \& \quad \underline{\int_a^b} cf\, d\alpha = c\underline{\int_a^b} f\, d\alpha$$

$$\because \overline{\int_a^b} cf\, d\alpha = \underline{\int_a^b} cf\, d\alpha \quad \text{and} \quad \overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha$$

$$\therefore \int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha \qquad \odot$$

ii) If $f_1 \in \boldsymbol{R}(\alpha)$ and $f_2 \in \boldsymbol{R}(\alpha)$ on $[a,b]$, then $f_1 + f_2 \in \boldsymbol{R}(\alpha)$ and

$$\int_a^b (f_1 + f_2)\,d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha .$$

Proof

If $f = f_1 + f_2$ and P is any partition of $[a,b]$, we have

$$m_i' + m_i'' \le m_i \le M_i \le M_i' + M_i''$$

where M_i', m_i', M_i'', m_i'' and M_i, m_i are the bounds of f_1, f_2 and f respectively in $[x_{i-1}, x_i]$.

Multiplying throughout by $\Delta\alpha_i$ and adding the inequalities for $i = 1, 2, ..., n$, we get

$$L(P, f_1,\alpha) + L(P, f_2,\alpha) \le L(P, f,\alpha) \le U(P, f,\alpha) \le U(P, f_1,\alpha) + U(P, f_2,\alpha) \quad ....(i)$$

Since $f_1 \in \boldsymbol{R}(\alpha)$ and $f_2 \in \boldsymbol{R}(\alpha)$ on $[a,b]$ therefore $\exists$ $\varepsilon > 0$ and there are partitions P_1 and P_2 such that

$$\left.\begin{aligned} U(P_1, f_1,\alpha) - L(P_1, f_1,\alpha) &< \varepsilon\\ \text{and}\quad U(P_2, f_2,\alpha) - L(P_2, f_2,\alpha) &< \varepsilon \end{aligned}\right\} \ ....\ (ii)$$

These inequalities hold if P_1 and P_2 are replaced by their common refinement P.

$(ii) \Rightarrow \left[U(P, f_1,\alpha)+U(P, f_2,\alpha)\right]-\left[L(P, f_1,\alpha)+L(P, f_2,\alpha)\right] < 2\varepsilon$

Using (i) we have

$$U(P, f,\alpha)-L(P, f,\alpha) < 2\varepsilon$$

which proves that $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$

With the same partition P, we have

$$U(P, f_1,\alpha) < \int_a^b f_1\, d\alpha + \varepsilon$$

and $$U(P, f_2,\alpha) < \int_a^b f_2\, d\alpha + \varepsilon$$

Hence (i) implies that

$$\int_a^b f\, d\alpha \le U(P, f,\alpha) < \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha + 2\varepsilon$$

$\because$ ε is arbitrary, we conclude that

$$\int_a^b f\, d\alpha \le \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$$

Similarly if we consider the lower sums we arrive at

$$\int_a^b f\, d\alpha \ge \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$$

Combining the above two results, we have

$$\int_a^b f\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$$ ⊙

iii) If $f_1(x) \le f_2(x)$ on $[a,b]$, then

$$\int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha$$

Proof

Let $f(x) \ge 0$, then $M_i \ge 0 \quad \Rightarrow U(P, f,\alpha) \ge 0$

and $\therefore \int_a^b f\, d\alpha \ge 0$

$$\because f_1 \le f_2 \qquad \therefore f_2 - f_1 \ge 0$$

$$\Rightarrow \int_a^b (f_2 - f_1)\, d\alpha \ge 0 \qquad \Rightarrow \int_a^b f_2\, d\alpha - \int_a^b f_1\, d\alpha \ge 0$$

$$\Rightarrow \int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha$$ ⊙

➢ ***Note***

(i) $(f+g)(x) = f(x)+g(x) \le \sup f + \sup g$

$\Rightarrow \sup(f+g) \le \sup f + \sup g$

(ii) $(f+g)(x) = f(x)+g(x) \ge \inf f + \inf g$

$\Rightarrow \inf(f+g) \ge \inf f + \inf g$

iv) If $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$ and if $a<c<b$, then $f \in \boldsymbol{R}(\alpha)$ on $[a,c]$ and on $[c,b]$ and

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

Proof

Since $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$, therefore for $\varepsilon > 0$, $\exists$ a partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

Let P^* be the refinement of P such that $P^* = P \cup \{c\}$

$$\therefore\ L(P,f,\alpha) \le L(P^*,f,\alpha) \le U(P^*,f,\alpha) \le U(P,f,\alpha) \ \ldots\ldots\ldots\ldots (i)$$

$$\Rightarrow\ U(P^*,f,\alpha) - L(P^*,f,\alpha) \le U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \ \ldots\ldots\ldots (ii)$$

Let P_1, P_2 denote the sets of points of P^* between $[a,c]$, $[c,b]$ respectively.

Clearly P_1, P_2 are partitions of $[a,c]$, $[c,b]$ respectively and $P^* = P_1 \cup P_2$.

Also $\quad U(P^*,f,\alpha) = U(P_1,f,\alpha) + U(P_2,f,\alpha) \ \ldots\ldots\ldots\ (iii)$

and $\quad L(P^*,f,\alpha) = L(P_1,f,\alpha) + L(P_2,f,\alpha) \ \ldots\ldots\ldots\ldots\ (iv)$

$$\therefore\ \{U(P_1,f,\alpha) - L(P_1,f,\alpha)\} + \{U(P_2,f,\alpha) - L(P_2,f,\alpha)\}$$
$$= U(P^*,f,\alpha) - L(P^*,f,\alpha) < \varepsilon$$

Since each bracket on the left is non-negative, it follows that

$$U(P_1,f,\alpha) - L(P_1,f,\alpha) < \varepsilon$$

and $\quad U(P_2,f,\alpha) - L(P_2,f,\alpha) < \varepsilon$

$\Rightarrow\ f \in \boldsymbol{R}(\alpha)$ on $[a,c]$ and on $[c,b]$.

We know that for any functions f_1 and f_2, if $f = f_1 + f_2$, then

$$\inf f \ \ge\ \inf f_1 + \inf f_2$$

and $\quad \sup f \ \le\ \sup f_1 + \sup f_2$

Now for any partitions P_1, P_2 of $[a,c]$, $[c,b]$ respectively, if $P^* = P_1 \cup P_2$, then

$$U(P^*,f,\alpha) = U(P_1,f,\alpha) + U(P_2,f,\alpha)$$

Hence on taking the infimum for all partitions, we get

$$\overline{\int_a^b} f\,d\alpha \ \ge\ \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha$$

But since $f \in \boldsymbol{R}(\alpha)$ on $[a,c]$, $[c,b]$, $[a,b]$

$$\therefore\ \int_a^b f\,d\alpha \ \ge\ \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \ \ldots\ldots\ldots\ldots\ (v)$$

Again $\quad L(P^*,f,\alpha) = L(P_1,f,\alpha) + L(P_2,f,\alpha)$

and on taking the supremum for all partitions, we get

$$\underline{\int_a^b} f\,d\alpha \ \le\ \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha$$

But since $f \in \boldsymbol{R}(\alpha)$ on $[a,c]$, $[c,b]$, $[a,b]$

$$\therefore\ \int_a^b f\,d\alpha \ \le\ \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \ \ldots\ldots\ldots\ldots\ldots (vi)$$

(v) and (vi) imply that

$$\int_a^b f\,d\alpha \ =\ \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

⊙

v) If $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$ and $\left| f(x) \right| \le M$ on $[a,b]$, then

$$\left| \int_a^b f\, d\alpha \right| \le M\left[\alpha(b)-\alpha(a)\right]$$

Proof

We know that

$$\int_a^b f\, d\alpha \le U\left(P,f,\alpha\right)$$

$$= \sum M_i\, \Delta\alpha_i \le M \sum \Delta\alpha_i$$

But

$$\sum \Delta\alpha_i = \alpha(b)-\alpha(a)$$

$$\Rightarrow \left| \int_a^b f\, d\alpha \right| \le M\left[\alpha(b)-\alpha(a)\right]$$ ⊙

vi) If $f \in \boldsymbol{R}(\alpha_1)$ and $f \in \boldsymbol{R}(\alpha_2)$, then $f \in \boldsymbol{R}(\alpha_1+\alpha_2)$ and

$$\int_a^b f\, d(\alpha_1+\alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$$

and if $f \in \boldsymbol{R}(\alpha)$ and c is a positive constant, then $f \in \boldsymbol{R}(c\alpha)$ and

$$\int_a^b f\, d(c\alpha) = c\int_a^b f\, d\alpha$$

Proof

Since $f \in \boldsymbol{R}(\alpha_1)$ and $f \in \boldsymbol{R}(\alpha_2)$, therefore for $\varepsilon > 0$, there exists partitions P_1, P_2 of $[a,b]$ such that

$$U\left(P_1,f,\alpha_1\right) - L\left(P_1,f,\alpha_1\right) < \frac{\varepsilon}{2}$$

$$\text{and} \quad U\left(P_2,f,\alpha_2\right) - L\left(P_2,f,\alpha_2\right) < \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$

$$\left.\begin{aligned} \therefore\ & U\left(P,f,\alpha_1\right) - L\left(P,f,\alpha_1\right) < \frac{\varepsilon}{2} \\ \&\ & U\left(P,f,\alpha_2\right) - L\left(P,f,\alpha_2\right) < \frac{\varepsilon}{2} \end{aligned}\right\} \ldots\ldots\ldots\ldots.. (i)$$

Let m_i , M_i be bounds of f in $\left[x_{i-1},x_i\right]$

Take $\alpha = \alpha_1 + \alpha_2$

$\Rightarrow \Delta\alpha_i = \Delta\alpha_{1i} + \Delta\alpha_{2i}$

$$\begin{aligned} \therefore\ U\left(P,f,\alpha\right) &= \sum M_i\, \Delta\alpha_i \\ &= \sum M_i\left(\Delta\alpha_{1i}+\Delta\alpha_{2i}\right) \\ &= U\left(P,f,\alpha_1\right) + U\left(P,f,\alpha_2\right) \end{aligned}$$

Similarly

$$L\left(P,f,\alpha\right) = L\left(P,f,\alpha_1\right) + L\left(P,f,\alpha_2\right)$$

$$\therefore\ U\left(P,f,\alpha\right) - L\left(P,f,\alpha\right) = U\left(P,f,\alpha_1\right) - L\left(P,f,\alpha_1\right) + U\left(P,f,\alpha_2\right) - L\left(P,f,\alpha_2\right)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{by } (i)$$

$\Rightarrow f \in \boldsymbol{R}(\alpha)$ where $\alpha = \alpha_1 + \alpha_2$

To prove the second part, we notice that

$$\int_a^b f\,d\alpha = \inf U(P,f,\alpha)$$

$$= \inf\{U(P,f,\alpha_1)+U(P,f,\alpha_2)\}$$

$$\geq \inf U(P,f,\alpha_1)+\inf U(P,f,\alpha_2)$$

$$= \int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2 \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Similarly by taking the supremum of lower sum of partition we arrive that

$$\int_a^b f\,d\alpha \leq \int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2 \quad \ldots\ldots\ldots\ldots\ldots (iii)$$

From (*ii*) and (*iii*)

$$\int_a^b f\,d\alpha = \int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2$$

i.e. $$\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2 \qquad \because\ \alpha=\alpha_1+\alpha_2$$

Now $\because$ $f\in \boldsymbol{R}(\alpha)$ $\therefore$ for $\varepsilon>0$, $\exists$ a partition P of $[a,b]$ such that

$$U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon \quad \ldots\ldots\ldots\ldots (iv)$$

Let $\alpha'=c\alpha$ then $\Delta\alpha_i'=\Delta(c\alpha_i)=c\Delta\alpha_i$

$$\Rightarrow\ U(P,f,\alpha')=\sum M_i\,\Delta\alpha_i'$$

$$=\sum M_i\,(c\Delta\alpha_i)$$

$$=c\sum M_i\,\Delta\alpha_i$$

$$=c\,U(P,f,\alpha)$$

Similarly, $L(P,f,\alpha')=c\,L(P,f,\alpha)$

$$\Rightarrow\ U(P,f,\alpha')-L(P,f,\alpha')=c\{U(P,f,\alpha)-L(P,f,\alpha)\} < c\varepsilon \qquad \text{by } (iv)$$

$\Rightarrow$ $f\in\boldsymbol{R}(\alpha')$ where $\alpha'=c\alpha$

Also $$\int_a^b f\,d\alpha'=\inf U(P,f,\alpha')$$

$$=\inf\ c\,U(P,f,\alpha)$$

$$=c\inf U(P,f,\alpha)$$

$$=c\int_a^b f\,d\alpha$$

and

$$\int_a^b f\,d\alpha'=\sup L(P,f,\alpha')$$

$$=\sup cU(P,f,\alpha)$$

$$=c\sup U(P,f,\alpha)$$

$$=c\int_a^b f\,d\alpha$$

Hence

$$\int_a^b f\,d\alpha'=c\int_a^b f\,d\alpha \quad \text{where } \alpha'=c\alpha$$ ⊙

> **Lemma**
> If M & m are the supremum and infimum of f and M', m' are the supremum & infimum of $|f|$ on $[a,b]$ then $M'-m' \leq M-m$.

Proof

Let $x_1, x_2 \in [a,b]$, then

$$\big||f(x_1)|-|f(x_2)|\big| \leq |f(x_1)-f(x_2)| \ \ldots\ldots\ldots\ldots\ldots\ldots(A)$$

$\because$ M and m denote the supremum and infimum of $f(x)$ on $[a,b]$

$\therefore$ $f(x) \leq M$ & $f(x) \geq m$ $\forall\ x \in [a,b]$

$\because$ $x_1, x_2 \in [a,b]$

$\therefore$ $f(x_1) \leq M$ and $f(x_2) \geq m$

$\Rightarrow$ $f(x_1) \leq M$ and $-f(x_2) \leq -m$

$\Rightarrow$ $f(x_1)-f(x_2) \leq M-m$ ……………... (i)

Interchanging x_1 & x_2, we get

$$-\left[f(x_1)-f(x_2)\right] \leq M-m \ \ldots\ldots\ldots\ldots\ (ii)$$

(i) & (ii) $\Rightarrow$ $|f(x_1)-f(x_2)| \leq M-m$

$\Rightarrow$ $\big||f(x_1)|-|f(x_2)|\big| \leq M-m$ by eq. (A) ………..(I)

$\because$ M' and m' denote the supremum and infimum of $|f(x)|$ on $[a,b]$

$\therefore$ $|f(x)| \leq M'$ and $|f(x)| \geq m'$ $\forall\ x \in [a,b]$

$\Rightarrow$ $\exists\ \varepsilon > 0$ such that

$$|f(x_1)| > M'-\varepsilon \ \ldots\ldots\ldots\ldots\ (iii)$$

and $|f(x_2)| < m'+\varepsilon$ $\Rightarrow$ $-|f(x_2)|+\varepsilon > -m'$ ………….. (iv)

From (iii) and (iv), we get

$$|f(x_1)|-|f(x_2)|+\varepsilon > M'-m'-\varepsilon$$

$$\Rightarrow\ 2\varepsilon+|f(x_1)|-|f(x_2)| > M'-m'$$

$\because$ ε is arbitrary $\therefore$ $M'-m' \leq |f(x_1)|-|f(x_2)|$ ……….…. (v)

Interchanging x_1 & x_2, we get

$$M'-m' \leq -\big(|f(x_1)|-|f(x_2)|\big) \ \ldots\ldots\ldots\ldots\ (vi)$$

Combining (v) and (vi), we get

$$M'-m' \leq \big||f(x_1)|-|f(x_2)|\big| \ \ldots\ldots\ldots\ldots\ (\text{II})$$

From (I) and (II), we have the require result

$$M'-m' \leq M-m$$

⊙

> **Theorem**
> If $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$, then $|f| \in \boldsymbol{R}(\alpha)$ on $[a,b]$ and $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$.

Proof

$\because$ $f \in \boldsymbol{R}(\alpha)$

$\therefore$ given $\varepsilon > 0$ $\exists$ a partition P of $[a,b]$ such that

$$U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon$$

i.e. $$\sum M_i\,\Delta\alpha_i-\sum m_i\,\Delta\alpha_i=\sum(M_i-m_i)\Delta\alpha_i < \varepsilon$$

Where M_i and m_i are supremum and infimum of f on $[x_{i-1},x_i]$

Now if M_i' and m_i' are supremum and infimum of $|f|$ on $[x_{i-1},x_i]$ then

$$M_i'-m_i' \leq M_i-m_i$$

$$\Rightarrow \sum\left(M_i' - m_i'\right)\Delta\alpha_i \le \sum\left(M_i - m_i\right)\Delta\alpha_i$$
$$\Rightarrow U\left(P,|f|,\alpha\right) - L\left(P,|f|,\alpha\right) \le U\left(P,f,\alpha\right) - L\left(P,f,\alpha\right) < \varepsilon$$
$$\Rightarrow |f| \in \boldsymbol{R}(\alpha).$$

Take $c = +1$ or -1 to make $c\int f\,d\alpha \ge 0$

Then $\left|\int_a^b f\,d\alpha\right| = c\int_a^b f\,d\alpha$ ……………. (i)

Also $c\,f(x) \le |f(x)| \quad \forall\ x \in [a,b]$

$$\Rightarrow \int_a^b c\,f\,d\alpha \le \int_a^b |f|\,d\alpha \Rightarrow c\int_a^b f\,d\alpha \le \int_a^b |f|\,d\alpha \text{ ………. } (ii)$$

From (i) and (ii), we have

$$\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$$

⊙

> **Theorem**
> If $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$, then $f^2 \in \boldsymbol{R}(\alpha)$ on $[a,b]$.

Proof

$\because\ f \in \boldsymbol{R}(\alpha) \quad \Rightarrow |f| \in \boldsymbol{R}(\alpha)$

$\Rightarrow |f(x)| < M \qquad \forall\ x \in [a,b]$

$\because\ f \in \boldsymbol{R}(\alpha)\ \ \therefore$ given $\varepsilon > 0$, $\exists$ a partition P of $[a,b]$ such that

$$U\left(P,f,\alpha\right) - L\left(P,f,\alpha\right) < \varepsilon/2M \text{ ………… } (i)$$

If M_i & m_i denote the sup. & inf. of f on $\left[x_{i-1},x_i\right]$ then M_i^2 & m_i^2 are the sup. & inf. of f^2 on $\left[x_{i-1},x_i\right]$.

$$\Rightarrow U\left(P,f^2,\alpha\right) - L\left(P,f^2,\alpha\right) = \sum\left(M_i^2 - m_i^2\right)\Delta\alpha_i$$
$$= \sum\left(M_i + m_i\right)\left(M_i - m_i\right)\Delta\alpha_i$$

$\because\ f(x) \le |f(x)| \le M \qquad \forall\ x \in [a,b]$

and $f^2 = |f|^2$

$\therefore\ M_i \le M$ & $m_i \le M$

$$\Rightarrow U\left(P,f^2,\alpha\right) - L\left(P,f^2,\alpha\right) \le \sum\left(M + M\right)\left(M_i - m_i\right)\Delta\alpha_i$$
$$= 2M\sum\left(M_i - m_i\right)\Delta\alpha_i$$
$$= 2M\left[U\left(P,f,\alpha\right) - L\left(P,f,\alpha\right)\right] < 2M\cdot\frac{\varepsilon}{2M} = \varepsilon$$

$\Rightarrow f^2 \in \boldsymbol{R}(\alpha)$

⊙

> **Corollary**
> If $f \in \boldsymbol{R}(\alpha)$ & $g \in \boldsymbol{R}(\alpha)$ on $[a,b]$ then $fg \in \boldsymbol{R}(\alpha)$ on $[a,b]$.

Proof

$\because\ f \in \boldsymbol{R}(\alpha)\ ,\quad g \in \boldsymbol{R}(\alpha)$

$\therefore\ f + g \in \boldsymbol{R}(\alpha)\ ,\quad f - g \in \boldsymbol{R}(\alpha)$

$\Rightarrow (f+g)^2 \in \boldsymbol{R}(\alpha)\ ,\ (f-g)^2 \in \boldsymbol{R}(\alpha)$

$\Rightarrow (f+g)^2 - (f-g)^2 \in \boldsymbol{R}(\alpha) \ \Rightarrow\ 4fg \in \boldsymbol{R}(\alpha)$

and ultimately

$$fg \in \boldsymbol{R}(\alpha) \text{ on } [a,b]$$

⊙

> ### ➢ *Theorem*
> Assume α increases monotonically and $\alpha' \in \boldsymbol{R}$ on $[a,b]$. Let f be bounded real function on $[a,b]$. Then $f \in \boldsymbol{R}(\alpha)$ iff $f\alpha' \in \boldsymbol{R}$. In that case
> $$\int_a^b f\,d\alpha = \int_a^b f(x)\cdot\alpha'(x)\,dx$$

Proof

$\because$ $\alpha' \in \boldsymbol{R}$ on $[a,b]$

$\therefore$ given $\varepsilon > 0$ $\exists$ a partition P of $[a,b]$ such that

$$U(P,\alpha') - L(P,\alpha') < \varepsilon \quad \ldots\ldots\ldots\ldots\ldots (i)$$

The Mean-value theorem furnishes point $t_i \in [x_{i-1}, x_i]$ such that

$$\begin{aligned}\Delta\alpha_i &= \alpha(x_i) - \alpha(x_{i-1})\\ &= \alpha'(t_i)\Delta x_i \quad \text{for } i = 1,2,\ldots,n \quad \ldots\ldots\ldots\ldots (ii)\end{aligned}$$

If $s_i \in [x_{i-1}, x_i]$, then form (i) we have

$$\left|\sum \alpha'(s_i)\Delta x_i - \sum \alpha'(t_i)\Delta x_i\right| < \varepsilon \qquad \text{| Previously proved at page 6}$$

$$\Rightarrow \sum \left|\alpha'(s_i) - \alpha'(t_i)\right|\Delta x_i < \varepsilon \quad \ldots\ldots\ldots\ldots\ldots (iii)$$

Put $M = \sup|f(x)|$ and consider

$$\begin{aligned}&\left|\sum f(s_i)\Delta\alpha_i - \sum f(s_i)\alpha'(s_i)\Delta x_i\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots (A)\\ &= \left|\sum f(s_i)\alpha'(t_i)\Delta x_i - \sum f(s_i)\alpha'(s_i)\Delta x_i\right| \qquad \text{by } (ii)\\ &= \left|\sum f(s_i)\big(\alpha'(t_i) - \alpha'(s_i)\big)\Delta x_i\right|\\ &\le \left|\sum M\big(\alpha'(t_i) - \alpha'(s_i)\big)\right|\Delta x_i\\ &\le M\varepsilon \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (iv) \qquad \text{by } (iii)\end{aligned}$$

$$\Rightarrow \sum f(s_i)\Delta\alpha_i \le \sum f(s_i)\alpha'(s_i)\Delta x_i + M\varepsilon \qquad \text{for all choices of } s_i \in [x_{i-1}, x_i]$$

$$\Rightarrow U(P,f,\alpha) \le U(P,f\alpha') + M\varepsilon$$

The same arguments leads from (A) to

$$U(P,f\alpha') \le U(P,f,\alpha) + M\varepsilon$$

Thus $\left|U(P,f,\alpha) - U(P,f\alpha')\right| \le M\varepsilon \quad \ldots\ldots\ldots\ldots\ldots (v)$

$\because$ (i) remains true if P is replaced by any refinement

$\therefore$ (v) also remains true

$$\Rightarrow \left|\overline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f(x)\alpha'(x)\,dx\right| \le M\varepsilon$$

$\because$ ε was arbitrary

$$\therefore \overline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f(x)\alpha'(x)\,dx \qquad \text{for any bounded } f.$$

Using the same argument, we can prove from (iv) by considering the infimum of $|f(x)|$ that

$$\underline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f(x)\alpha'(x)\,dx$$

Hence

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha \iff \overline{\int_a^b} f(x)\alpha'(x)\,dx = \underline{\int_a^b} f(x)\alpha'(x)\,dx$$

Equivalently $f \in \boldsymbol{R}(\alpha) \iff f\alpha' \in \boldsymbol{R}(\alpha)$. ⊙

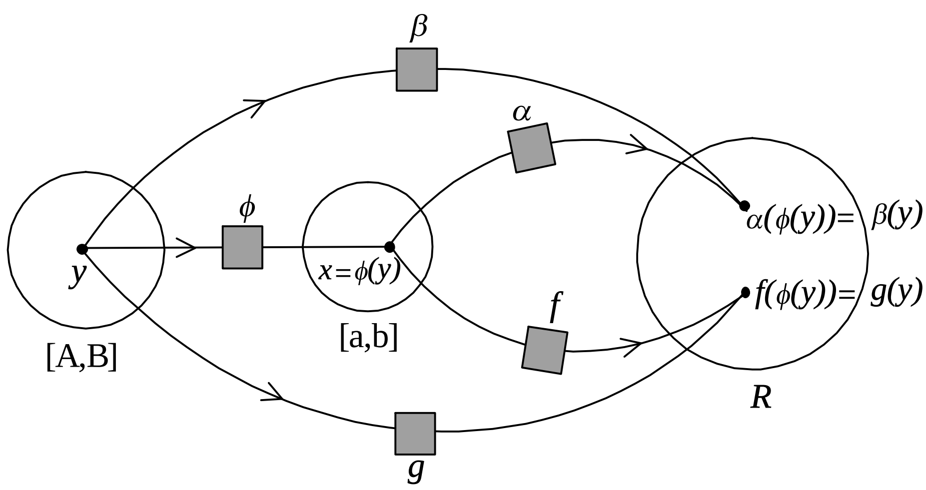
➢ *Theorem (Change of Variable)*

Suppose φ is a strictly increasing continuous function that maps an interval $[A,B]$ onto $[a,b]$. Suppose α is monotonically increasing on $[a,b]$ and $f \in \boldsymbol{R}(\alpha)$ on $[a,b]$. Define β and g on $[A,B]$ by

$$\beta(y) = \alpha(\varphi(y)) \quad , \quad g(y) = f(\varphi(y))$$

then $g \in \boldsymbol{R}(\beta)$ and $\int_A^B g\, d\beta = \int_a^b f\, d\alpha$.

Proof



To each partition $P = \{x_0, \ldots\ldots, x_n\}$ of $[a,b]$ corresponds a partition $Q = \{y_0, \ldots\ldots, y_n\}$ of $[A,B]$ because φ maps $[A,B]$ onto $[a,b]$.

$$\Rightarrow \; x_i = \varphi(y_i)$$

All partitions of $[A,B]$ are obtained in this way.

$\because$ The value taken by f on $[x_{i-1}, x_i]$ are exactly the same as those taken by g on $[y_{i-1}, y_i]$, we see that

$$U(Q,g,\beta) = U(P,f,\alpha)$$

and $$L(Q,g,\beta) = L(P,f,\alpha)$$

$\because$ $f \in R(\alpha)$ on $[a,b]$

$\therefore$ given $\varepsilon > 0$, we have

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

$$\Rightarrow U(Q,g,\beta) - L(Q,g,\beta) < \varepsilon$$

$\Rightarrow$ $g \in \boldsymbol{R}(\beta)$ and $\int_A^B g\, d\beta = \int_a^b f\, d\alpha$ ⊙

INTEGRATION AND DIFFERENTIATION

> **Theorem (Ist Fundamental Theorem of Calculus)**
>
> Let $f \in \boldsymbol{R}$ on $[a,b]$. For $a \le x \le b$, put $F(x) = \int\limits_a^x f(t)\,dt$, then F is continuous on $[a,b]$; furthermore, if f is continuous at point x_0 of $[a,b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

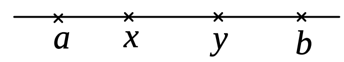
Proof

$\because \; f \in \boldsymbol{R}$

$\therefore \; f$ is bounded.

Let $|f(t)| \le M$ for $t \in [a,b]$

If $a \le x < y \le b$, then



$$\begin{aligned}
|F(y)-F(x)| &= \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right| \\
&= \left| \int_a^x f(t)\,dt + \int_x^y f(t)\,dt - \int_a^x f(t)\,dt \right| \\
&= \left| \int_x^y f(t)\,dt \right| \le \int_x^y |f(t)|\,dt \le M\int_x^y dt = M(y-x)
\end{aligned}$$

$\Rightarrow \; |F(y)-F(x)| < \varepsilon$ for $\varepsilon > 0$ provided $M|y-x| < \varepsilon$

i.e. $|F(y)-F(x)| < \varepsilon$ whenever $|y-x| < \dfrac{\varepsilon}{M}$

This proves the continuity (and, in fact, uniform continuity) of F on $[a,b]$.

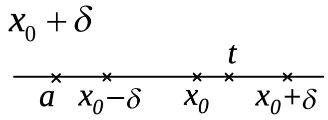
Next, we have to prove that if f is continuous at $x_0 \in [a,b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$

$$\text{i.e.} \quad \lim_{t \to x_0} \frac{F(t)-F(x_0)}{t-x_0} = f(x_0)$$

Suppose f is continuous at x_0. Given $\varepsilon > 0$, $\exists \; \delta > 0$ such that

$$|f(t)-f(x_0)| < \varepsilon \quad \text{if} \quad |t-x_0| < \delta \quad \text{where } t \in [a,b]$$

$$\Rightarrow \; f(x_0)-\varepsilon < f(t) < f(x_0)+\varepsilon \quad \text{if} \quad x_0-\delta < t < x_0+\delta$$



$$\Rightarrow \; \int_{x_0}^{t} \big(f(x_0)-\varepsilon\big)dt < \int_{x_0}^{t} f(t)\,dt < \int_{x_0}^{t} \big(f(x_0)+\varepsilon\big)dt$$

$$\Rightarrow \; \big(f(x_0)-\varepsilon\big)\int_{x_0}^{t} dt < \int_{x_0}^{t} f(t)\,dt < \big(f(x_0)+\varepsilon\big)\int_{x_0}^{t} dt$$

$$\Rightarrow \; \big(f(x_0)-\varepsilon\big)(t-x_0) < F(t)-F(x_0) < \big(f(x_0)+\varepsilon\big)(t-x_0)$$

$$\Rightarrow \; f(x_0)-\varepsilon < \frac{F(t)-F(x_0)}{t-x_0} < f(x_0)+\varepsilon$$

$$\Rightarrow \; \left| \frac{F(t)-F(x_0)}{t-x_0} - f(x_0) \right| < \varepsilon$$

$$\Rightarrow \; \lim_{t \to x_0} \frac{F(t)-F(x_0)}{t-x_0} = f(x_0)$$

$$\Rightarrow \; F'(x_0) = f(x_0)$$

⊙

℘ -------------------------------- ℘

> ➢ ***Theorem (IInd Fundamental Theorem of Calculus)***
>
> If $f \in \boldsymbol{R}$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then
>
> $$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof

$\because$ $f \in \boldsymbol{R}$ on $[a,b]$

$\therefore$ given $\varepsilon > 0$, $\exists$ a partition P of $[a,b]$ such that

$$U(P,f) - L(P,f) < \varepsilon$$

$\because$ F is differentiable on $[a,b]$

$\therefore$ $\exists$ $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(t_i)\Delta x_i$$

$$\Rightarrow F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i \quad \text{for } i = 1, 2, \ldots., n \quad \because F' = f$$

$$\Rightarrow \sum_{i=1}^{n} f(t_i)\Delta x_i = F(b) - F(a)$$

$$\Rightarrow \left| F(b) - F(a) - \int_a^b f(x)\,dx \right| < \varepsilon$$

$\boxed{\because \text{ if } f \in \boldsymbol{R}(\alpha) \text{ then } \left| \sum f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \varepsilon}$

$\because$ ε is arbitrary

$$\therefore \int_a^b f(x)\,dx = F(b) - F(a)$$ ⊙

> ➢ ***Theorem (Integration by Parts)***
>
> Suppose F and G are differentiable function on $[a,b]$, $F' = f \in \boldsymbol{R}$ and $G' = g \in \boldsymbol{R}$ then
>
> $$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx$$

Proof

Put $H(x) = F(x)G(x)$

$\Rightarrow$ $H' = F'(x)G(x) + F(x)G'(x) = h$

Now $\because$ $H \in \boldsymbol{R}$ and $h \in \boldsymbol{R}$ on $[a,b]$

$\therefore$ By applying the fundamental theorem of calculus to H and its derivative h, we have

$$\int_a^b h\,dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b \left[F'(x)G(x) + F(x)G'(x)\right]dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b f(x)G(x)\,dx + \int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a)$$

$$\Rightarrow \int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx$$ ⊙

ॐ --------------------------------- ଓ

➢ ***Question***

Show that the function f defined on $[0,1]$ by

$$f(x)=\begin{cases}1 & ;\ x \text{ is rational}\\ 0 & ;\ x \text{ is irrational}\end{cases}$$

is not integrable on $[0,1]$

Solution

For any partition P of $[0,1]$, $m_k=0$, $M_k=1$

$$\Rightarrow\ S(P,f)=\sum_{k=1}^{n}M_k\,\Delta x_k=\sum_{k=1}^{n}\Delta x_k=1-0=1$$

and $\quad L(P,f)=\sum_{k=1}^{n}m_k\,\Delta x_k=0$

so that $\quad \overline{\int_0^1} f\,dx=1 \quad , \quad \underline{\int_0^1} f\,dx=0$

i.e. $\quad \overline{\int_0^1} f\,dx\neq\underline{\int_0^1} f\,dx \quad \Rightarrow f$ is not integrable on $[0,1]$. ⊙

➢ ***Question***

Show that $f(x)=\sin x$ is Riemann integrable over $\left[0,\dfrac{\pi}{2}\right]$.

Solution

Take $P=\left\{0,\dfrac{\pi}{2n},\dfrac{\pi}{n},\dfrac{3\pi}{2n},.....,\dfrac{n\pi}{2n}\right\}$ by dividing $\left[0,\dfrac{\pi}{2}\right]$ into n equal parts.

Then $M_k=\sin\dfrac{k\pi}{2n}$, $m_k=\sin\dfrac{(k-1)\pi}{2n}$

$$\Rightarrow\ S(P,f)-L(P,f)=\sum\left(\sin\frac{k\pi}{2n}-\sin\frac{(k-1)\pi}{2n}\right)\frac{\pi}{2n}$$

$$\leq\frac{\pi}{2n}\ <\varepsilon \qquad \text{for } n>n_0=\frac{\pi}{2\varepsilon}$$

$\Rightarrow f$ is Riemann integrable over $\left[0,\dfrac{\pi}{2}\right]$. ⊙

➢ ***Question***

Show that $f(x)=\begin{cases}1/x & ;\ x \text{ is rational},\ 0<x\leq 1\\ 0 & ;\ x \text{ is irrational}\end{cases}$

is integrable on $[0,1]$.

Solution

f is continuous at each irrational. And rational numbers are dense in $[0,1]$.

Also $L(P,f)=0$ for any partition P of $[0,1]$ so that $\underline{\int_0^1} f\,dx=0$

$\because\ f\geq 0 \quad \therefore\ S(P,f)\geq 0 \quad \Rightarrow \overline{\int_0^1} f\,d\alpha\geq 0$ ………… (i)

$\because$ There are only finite number of points $\dfrac{p}{q}$ (rationals) for which $f\left(\dfrac{p}{q}\right) = \dfrac{q}{p} \geq \dfrac{\varepsilon}{2}$

$\therefore$ Suppose $f(x) \geq \dfrac{\varepsilon}{2}$ for k values of x in $[0,1]$

Take P_1 such that $|P_1| < \dfrac{\varepsilon}{2k}$.

Consider $S(P_1, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$

There are at most k values for which $\dfrac{\varepsilon}{2} \leq M_i \leq 1$. For all other values $M_i > \dfrac{\varepsilon}{2}$.

$$\Rightarrow S(P_1, f) = \sum_{k \text{ values}} M_i(x_i - x_{i-1}) + \sum_{\textit{other} \text{ values}} M_i(x_i - x_{i-1})$$

$$\leq \frac{\varepsilon}{2k} \cdot k + \frac{\varepsilon}{2} \sum (x_i - x_{i-1}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\because$ ε is arbitrary

$\therefore$ $S(P_1, f) \leq 0$ and $\overline{\int_0^1} f\,dx \leq 0$ ………… (ii)

By (i) and (ii), we have

$$\overline{\int_0^1} f\,dx = 0$$

Hence $\int_0^1 f\,dx = 0$ ⊙

➢ *Note*

If f is integrable then $|f|$ is also integrable but the converse is false.

For example, let f be a function defined on $[a,b]$ by

$$f(x) = \begin{cases} 1 & ; \ x \in \mathbb{Q} \cap [a,b] \\ -1 & ; \ \text{otherwise} \end{cases}$$

Then $|f|$ is Riemann-integrable but f is not.

Chapter 7 – Functions of Bounded Variation.

We shall now discuss the concept of functions of bounded variation which is closely associated to the concept of monotonic functions and has wide application in mathematics. These functions are used in Riemann-Stieltjes integrals and Fourier series.

Let a function f be defined on an interval $[a,b]$ and $P=\{a=x_0,x_1,........,x_n=b\}$ be a partition of $[a,b]$. Consider the sum $\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$. The set of these sums is infinite. It changes when we make a refinement in a partition. If this set of sums is bounded above then the function f is said to be a *bounded variation* and the supremum of the set is called the *total variation* of the function f on $[a,b]$, and is denoted by $V(f;a,b)$ or $V_f(a,b)$ and it is also affiliated as $V(f)$ or V_f.

Thus

$$V(f;a,b)=\sup\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$$

The supremum being taken over all the partition of $[a,b]$.

Hence the function f is said to be of *bounded variation* on $[a,b]$ if, and only, if its total variation is finite i.e. $V(f;a,b)<\infty$.

Note

Since for $x\le c\le y$, we have

$$|f(y)-f(x)|\le|f(y)-f(c)|+|f(c)-f(x)|$$

Therefore the sum $\sum|f(x_i)-f(x_{i-1})|$ can not be decrease (it can, in fact only increase) by the refinement of the partition.

Theorem

A bounded monotonic function is a function of bounded variation.

Proof

Suppose a function f is monotonically increasing on $[a,b]$ and P is any partition of $[a,b]$ then

$$\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})| = \sum_{i=1}^{n}\left(f(x_i)-f(x_{i-1})\right) = f(b)-f(a)$$

$$\therefore\ V(f;a,b)=\sup\sum|f(x_i)-f(x_{i-1})| = f(b)-f(a)\ \text{(finite)}$$

Hence the function f is of bounded variation on $[a,b]$.

Similarly a monotonically decreasing bounded function is of bounded variation with total variation $=f(a)-f(b)$.

Thus for a bounded monotonic function f

$$V(f)=|f(b)-f(a)|$$ ❑

❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀❀

Example

A continuous function may not be a function of bounded variation.

e.g. Consider a function f, where

$$f(x)\begin{cases} x\sin\dfrac{\pi}{x} & ;\ \text{when } 0<x\le 1 \\ 0 & ;\ \text{when } x=0 \end{cases}$$

It is clear that f is continuous on $[0,1]$.

Let us choose the partition $P=\left\{0,\dfrac{2}{2n+1},\dfrac{2}{2n-1},........,\dfrac{2}{5},\dfrac{2}{3},1\right\}$

Then

$$\sum\left|f(x_i)-f(x_{i-1})\right|=\left|f(1)-f\left(\frac{2}{3}\right)\right|+\left|f\left(\frac{2}{3}\right)-f\left(\frac{2}{5}\right)\right|+.......+\left|f\left(\frac{2}{2n+1}\right)-f(0)\right|$$

$$=\left|\sin\pi-\frac{2}{3}\sin\left(\frac{3\pi}{2}\right)\right|+\left|\frac{2}{3}\sin\left(\frac{3\pi}{2}\right)-\frac{2}{5}\sin\left(\frac{5\pi}{2}\right)\right|+.........$$

$$.....+\left|\frac{2}{2n+1}\sin\left(\frac{(2n+1)\pi}{2}\right)-0\right|$$

$$=\frac{2}{3}+\left(\frac{2}{3}+\frac{2}{5}\right)+\left(\frac{2}{5}+\frac{2}{7}\right)+...........+\left(\frac{2}{2n-1}+\frac{2}{2n+1}\right)+\frac{2}{2n+1}$$

$$=\left(2\left(\frac{2}{3}\right)+2\left(\frac{2}{5}\right)+2\left(\frac{2}{7}\right)+............+2\left(\frac{2}{2n+1}\right)\right)$$

$$=4\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+............+\frac{1}{2n+1}\right)$$

Since the infinite series $\dfrac{1}{3}+\dfrac{1}{5}+\dfrac{1}{7}+..........$ is divergent, therefore its partial sums sequence $\{S_n\}$, where $S_n=\dfrac{1}{3}+\dfrac{1}{5}+\dfrac{1}{7}+........+\dfrac{1}{2n+1}$, is not bounded above.

Thus $\sum\left|f(x_i)-f(x_{i-1})\right|$ can be made arbitrarily large by taking n sufficiently large.

$\Rightarrow V(f;0,1)\to\infty$ and so f is not of bounded variation. ❑

Remarks

A function of bounded variation is not necessarily continuous.

e.g. the step-function $f(x)=[x]$, where $[x]$ denotes the greatest integer not greater than x, is a function of bounded variation on $[0,2]$ but is not continuous.

Theorem

If the derivative of the function f exists and is bounded on $[a,b]$, then f is of bounded variation on $[a,b]$.

Proof

$\because$ f' is bounded on $[a,b]$

$\therefore$ $\exists$ k such that $\left|f'(x)\right|\le k$ $\forall$ $x\in[a,b]$.

Let P be any partition of the interval $[a,b]$ then

$$\sum\left|f(x_i)-f(x_{i-1})\right|=\sum\left|x_i-x_{i-1}\right|f'(c)\ ,\quad c\in[a,b]\quad \text{(by M.V.T)}$$

$$\le k\left|b-a\right|$$

$\Rightarrow V(f;a,b)$ is finite. $\Rightarrow$ f is of bounded variation. ❑

Note

Boundedness of f' is a sufficient condition for $V(f)$ to be finite and is not necessary.

Theorem

A function of bounded variation is necessarily bounded.

Proof

Suppose f is of bounded variation on $[a,b]$.

For any $x\in[a,b]$, consider the partition $\{a,x,b\}$, consisting of just three points then

$$|f(x)-f(a)|+|f(b)-f(x)| \le V(f;a,b)$$

$$\Rightarrow |f(x)-f(a)| \le V(f;a,b)$$

Again

$$\begin{aligned}|f(x)| &= |f(a)+f(x)-f(a)|\\ &\le |f(a)|+|f(x)-f(a)|\\ &\le |f(a)|+V\big(f;a,b\big) < \infty\end{aligned}$$

$\Rightarrow$ f is bounded on $[a,b]$. ❑

Properties of functions of bounded variation

1) The sum (difference) of two functions of bounded variation is also of bounded variation.

Proof

Let f and g be two functions of bounded variation on $[a,b]$. Then for any partition P of $[a,b]$ we have

$$\begin{aligned}\sum\big|(f+g)(x_i)-(f+g)(x_{i-1})\big| &= \sum\big|\{f(x_i)+g(x_i)\}-\{f(x_{i-1})+g(x_{i-1})\}\big|\\ &= \sum\big|f(x_i)-f(x_{i-1})+g(x_i)-g(x_{i-1})\big|\\ &\le \sum\big|f(x_i)-f(x_{i-1})\big|+\sum\big|g(x_i)-g(x_{i-1})\big|\\ &\le V(f;a,b)+V(g;a,b)\end{aligned}$$

$$\Rightarrow V\big(f+g;a,b\big) \le V(f;a,b)+V(g;a,b)$$

This show that the function $f+g$ is of bounded variation.

Similarly it can be shown that $f-g$ is also of bounded variation.

i.e. $V(f-g) \le V(f)+V(g)$ ❑

Note

(i) If f and g are monotonic increasing on $[a,b]$ then $(f-g)$ is of bounded variation on $[a,b]$.

(ii) If c is constant, the sums $\sum|f(x_i)-f(x_{i-1})|$ and therefore the total variation function, $V(f)$ is same for f and $f-c$.

2) The product of two functions of bounded variation is also of bounded variation.

Proof

Let f and g be two functions of bounded variation on $[a,b]$.

$\Rightarrow$ f and g are bounded and $\exists$ a number k such that

$$|f(x)| \le k \quad \& \quad |g(x)| \le k \qquad \forall\ x\in[a,b].$$

For any partition P of $[a,b]$ we have

$$\begin{aligned}
&\sum\left|(fg)(x_i)-(fg)(x_{i-1})\right| \\
&=\sum\left|f(x_i)g(x_i)-f(x_{i-1})g(x_{i-1})\right| \\
&=\sum\left|f(x_i)g(x_i)-f(x_i)g(x_{i-1})+f(x_i)g(x_{i-1})-f(x_{i-1})g(x_{i-1})\right| \\
&=\sum\left|f(x_i)\{g(x_i)-g(x_{i-1})\}+g(x_{i-1})\{f(x_i)-f(x_{i-1})\}\right| \\
&\le\sum\left|f(x_i)\right|\left|g(x_i)-g(x_{i-1})\right|+\sum\left|g(x_{i-1})\right|\left|f(x_i)-f(x_{i-1})\right| \\
&\le k\sum\left|g(x_i)-g(x_{i-1})\right|+k\sum\left|f(x_i)-f(x_{i-1})\right| \\
&\le k\,V(g)+k\,V(f)
\end{aligned}$$

$\Rightarrow$ fg is of bounded variation on $[a,b]$. ❑

Note

Theorems like the above, could not be applied to quotients of functions because the reciprocal of a function of bounded variation need not be of bounded variation.

e.g. if $f(x)\to 0$ as $x\to x_0$, then $\dfrac{1}{f(x)}$ will not be bounded and therefore can not be of bounded variation on any interval which contains x_0.

Therefore to consider quotient, we avoid functions whose values becomes arbitrarily close to zero.

3) If f is a function of bounded variation on $[a,b]$ and if $\exists$ a positive number k such that $|f(x)| \ge k \quad \forall\ x\in[a,b]$ then $\dfrac{1}{f}$ is also of bounded variation on $[a,b]$.

Proof

For any partition P of $[a,b]$, we have

$$\begin{aligned}
\sum\left|\frac{1}{f}(x_i)-\frac{1}{f}(x_{i-1})\right| &= \sum\left|\frac{1}{f(x_i)}-\frac{1}{f(x_{i-1})}\right| \\
&=\sum\left|\frac{f(x_{i-1})-f(x_i)}{f(x_i)f(x_{i-1})}\right| \\
&\le\frac{1}{k^2}\sum\left|f(x_{i-1})-f(x_i)\right| \le \frac{1}{k^2}V(f;a,b)
\end{aligned}$$

$\Rightarrow$ $\dfrac{1}{f}$ is of bounded variation on $[a,b]$. ❑

4) If f is of bounded variation on $[a,b]$, then it is also of bounded variation on $[a,c]$ and $[c,b]$, where c is a point of $[a,b]$, and conversely. Also

$$V(f;a,b)=V(f;a,c)+V(f;c,b).$$

Proof

a) Let, first, f be of bounded variation on $[a,b]$.

Take $P_1=\{a=x_0,x_1,.....,x_m=c\}$ & $P_2=\{c=y_0,y_1,.....,y_n=b\}$ any two partitions of $[a,c]$ and $[c,b]$ respectively.

Evidently, $P=P_1\cup P_2=\{a=x_0,....,x_m,y_0,....,y_n=b\}$ is a partition of $[a,b]$.

We have

$$\left\{\sum_{i=1}^{m}\left|f(x_i)-f(x_{i-1})\right|+\sum_{i=1}^{n}\left|f(y_i)-f(y_{i-1})\right|\right\} \le V(f;a,b)$$

$$\Rightarrow \sum_{i=1}^{m} \left| f(x_i) - f(x_{i-1}) \right| \le V\left(f;a,b\right)$$

and $$\sum_{i=1}^{n} \left| f(y_i) - f(y_{i-1}) \right| \le V\left(f;a,b\right)$$

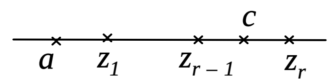
$\Rightarrow$ f is of bounded variation on $[a,c]$ and $[c,b]$ both.

b) Let, now, f be of bounded variation on $[a,c]$ and $[c,b]$ both.

Let $P = \{a = z_0, z_1, \ldots\ldots, z_n = b\}$ be a partition of $[a,b]$.

If it does not contain the point c, let us consider the partition $P^* = P \cup \{c\}$

Let $c \in [z_{r-1}, z_r]$ i.e. $z_{r-1} \le c \le z_r$, $\quad r < n$



Then

$$\sum_{i=1}^{n} \left| f(z_i) - f(z_{i-1}) \right| = \sum_{i=1}^{r-1} \left| f(z_i) - f(z_{i-1}) \right| + \left| f(z_r) - f(z_{r-1}) \right| + \sum_{i=r+1}^{n} \left| f(z_i) - f(z_{i-1}) \right|$$

$$\le \sum_{i=1}^{r-1} \left| f(z_i) - f(z_{i-1}) \right| + \left| f(c) - f(z_{r-1}) \right|$$

$$+ \left| f(z_r) - f(c) \right| + \sum_{i=r+1}^{n} \left| f(z_i) - f(z_{i-1}) \right|$$

$$\le V\left(f;a,c\right) + V\left(f;c,b\right)$$

$\Rightarrow$ f is of bounded variation on $[a,b]$ if it is of bounded variation on $[a,c]$ & $[c,b]$ both, then

$$V\left(f;a,b\right) \le V\left(f;a,c\right) + V\left(f;c,b\right) \quad \ldots\ldots\ldots\ldots (i)$$

Now let $\varepsilon > 0$ be any arbitrary number.

Since $V\left(f;a,c\right)$ and $V\left(f;c,b\right)$ are the total variation of f on $[a,c]$ & $[c,b]$ respectively therefore $\exists$ partition $P_1 = \{a = x_0, x_1, x_2, \ldots\ldots, x_m = c\}$ and $P_2 = \{c = y_0, y_1, y_3, \ldots\ldots, y_n = b\}$ of $[a,c]$ & $[c,b]$ respectively such that

$$\sum_{i=1}^{m} \left| f(x_i) - f(x_{i-1}) \right| > V\left(f;a,c\right) - \frac{\varepsilon}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

& $$\sum_{i=1}^{n} \left| f(y_i) - f(y_{i-1}) \right| > V\left(f;c,b\right) - \frac{\varepsilon}{2} \quad \ldots\ldots\ldots\ldots\ldots (iii)$$

Adding (ii) and (iii) we get

$$\sum_{i=1}^{m} \left| f(x_i) - f(x_{i-1}) \right| + \sum_{i=1}^{n} \left| f(y_i) - f(y_{i-1}) \right| > V\left(f;a,c\right) + V\left(f;c,b\right) - \varepsilon$$

$$\Rightarrow V\left(f;a,b\right) > V\left(f;a,c\right) + V\left(f;c,b\right) - \varepsilon$$

But ε is arbitrary positive number therefore we get

$$V\left(f;a,b\right) \ge V\left(f;a,c\right) + V\left(f;c,b\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

From (i) and (iv), we get

$$V\left(f;a,b\right) = V\left(f;a,c\right) + V\left(f;c,b\right)$$ ❑

Variation Function

Let f be a function of bounded variation on $[a,b]$ and x is a point of $[a,b]$. Then the total variation of f is $V(f;a,x)$ on $[a,x]$, which is clearly a function of x, is called the *total variation function* or simply the *variation function* of f and is denoted by $V_f(x)$, and when there is no scope for confusion, it is simply written as $V(x)$.

Thus $V_f(x) = V(f;a,x) \quad ; \quad (a \le x \le b)$

If x_1, x_2 are two points of the interval $[a,b]$ such that $x_2 > x_1$, then

$$\begin{aligned} 0 \le \left| f(x_2) - f(x_1) \right| &\le V(f;x_1,x_2) \\ &= V(f;a,x_1) - V(f;a,x_2) \\ &= V_f(x_2) - V_f(x_1) \end{aligned}$$

$$\Rightarrow V_f(x_2) \ge V_f(x_1)$$

implies that the variation function is monotonically increasing function on $[a,b]$.

CHARACTERIZATION OF FUNCTIONS OF BOUNDED VARIATION

Theorem

A function of bounded variation is expressible as the difference of two monotonically increasing function.

Proof

We have

$$\begin{aligned} f(x) &= \frac{1}{2}\big(V(x) + f(x)\big) - \frac{1}{2}\big(V(x) - f(x)\big) \\ &= G(x) - H(x) \qquad \text{(say)} \end{aligned}$$

We shall prove that these two functions $G(x)$ and $H(x)$ are monotonically increasing on $[a,b]$.

Now, if $x_2 > x_1$, we have

$$\begin{aligned} G(x_2) - G(x_1) &= \frac{1}{2}\big[V(x_2) - V(x_1) + f(x_2) - f(x_1)\big] \\ &= \frac{1}{2}\Big[V(f;x_1,x_2) - \big(f(x_1) - f(x_2)\big)\Big] \end{aligned}$$

Since $V(f;x_1,x_2) \ge f(x_1) - f(x_2)$

$$\Rightarrow G(x_2) - G(x_1) \ge 0 \qquad \text{i.e.} \quad G(x_2) \ge G(x_1)$$

so that the function $G(x)$ is monotonically increasing on $[a,b]$.

Again, we have

$$\begin{aligned} H(x_2) - H(x_1) &= \frac{1}{2}\Big[\big(V(x_2) - V(x_1)\big) - \big(f(x_2) - f(x_1)\big)\Big] \\ &= \frac{1}{2}\Big[V(f;x_1,x_2) - \big(f(x_2) - f(x_1)\big)\Big] \end{aligned}$$

so that as before

$$H(x_2) - H(x_1) \ge 0 \quad \text{i.e. } H(x_2) \ge H(x_1).$$

i.e. $H(x)$ is also monotonically increasing function.

Hence the result. ❑

Note

A function $f(x)$ is of bounded variation over the interval $[a,b]$ iff it can be expressed as the difference of two monotonically functions.

Theorem

Let f be of bounded variation on $[a,b]$. Let V be defined on $[a,b]$ as follows:

$$V(x) = V_f(x) = V\big(f; a, x\big) \quad \text{if} \quad a < x \le b \ , \ V(a) = 0 .$$

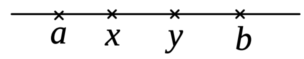
Then

i) V is an increasing function on $[a,b]$.

ii) $\big(V - f\big)$ is an increasing function on $[a,b]$.

Proof

If $a < x < y \le b$, we can write



$$V(f; a, y) = V(f; a, x) - V(f; x, y)$$

$$\Rightarrow V(y) - V(x) = V(f; x, y)$$

$$\because V(f; x, y) \ge 0$$

$$\therefore V(y) - V(x) \ge 0 \quad \Rightarrow V(x) \le V(y) \quad \text{and } (i) \text{ holds.}$$

To prove (*ii*), let $D(x) = V(x) - f(x)$ if $x \in [a,b]$.

Then, if $a \le x < y \le b$, we have

$$D(y) - D(x) = \big[V(y) - V(x)\big] - \big[f(y) - f(x)\big]$$

$$= V(f; x, y) - \big[f(y) - f(x)\big]$$

But from the definition of $V(f; x, y)$, it follows that

$$f(y) - f(x) \le V(f; x, y)$$

This means that $D(y) - D(x) \ge 0$ and (*ii*) holds. ❑

Theorem

If c be any point of $[a,b]$, then $V(x)$ is continuous at c if and only if $f(x)$ is continuous at c.

i.e. A point of continuity of $f(x)$ is also a point of continuity of $V(x)$ and conversely.

Proof

Firstly suppose that $V(x)$ is continuous at c.

Let $\varepsilon > 0$ be given, then $\exists \ \delta > 0$ such that

$$\big|V(x) - V(c)\big| < \varepsilon \quad \text{for} \quad \big|x - c\big| < \delta \quad \text{………..} \ (i)$$

Also, we have

$$\big|f(x) - f(c)\big| \le V(x) - V(c) \quad \text{if} \quad x > c \quad \text{……………} \ (ii)$$

And

$$\big|f(x) - f(c)\big| \le V(c) - V(x) \quad \text{if} \quad x < c \quad \text{……………} \ (iii)$$

From (*i*), (*ii*) and (*iii*), we deduce that

$$\big|f(x) - f(c)\big| \le \big|V(x) - V(c)\big| < \varepsilon \quad \text{for} \quad \big|x - c\big| < \delta$$

Which shows that $f(x)$ is continuous at c.

Now suppose that c is a point of continuity of $f(x)$ and let $\varepsilon > 0$ be given, then $\exists \ \delta > 0$ such that

$$\big|f(x) - f(c)\big| < \frac{\varepsilon}{2} \quad \text{for} \quad \big|x - c\big| < \delta$$

Also $\exists$ a partition $P = \big\{c = y_0, y_1, \ldots\ldots, y_{q-1}, y_q, \ldots\ldots, y_n = b\big\}$ of $[c,b]$ such that

$$\sum_{q=1}^{n} \big|f(y_q) - f(y_{q-1})\big| > V(f; c, b) - \frac{1}{2}\varepsilon \quad \text{……………} \ (iv)$$

Since as a result of introducing addition points to the partition P, the corresponding sum of the moduli of the differences of the function values at end points will not be decreased, therefore we may assume that

$$0 < y_1 - c < \delta$$

so that $\quad \left| f(y_1) - f(c) \right| < \dfrac{\varepsilon}{2}$ …………….. (v)

Thus (iv) becomes

$$V(f;c,b) - \frac{1}{2}\varepsilon < \frac{1}{2}\varepsilon + \sum_{q=2}^{n} \left| f(y_q) - f(y_{q-1}) \right| < \frac{1}{2}\varepsilon + V(f;y_1,b)$$

$$\Rightarrow V(f;c,b) - V(f;y_1,b) < \varepsilon$$

$$\Rightarrow V(y_1) - V(c) < \varepsilon$$

Thus for $0 < y_1 - c < \delta$, we have $0 < V(y_1) - V(c) < \varepsilon$

$$\therefore \lim_{x\to c+0} V(x) = V(c)$$

Similarly, we can have

$$\lim_{x\to c-0} V(x) = V(c)$$

Which shows that $V(x)$ is continuous at c. ❑

Note

$V(x)$ is continuous in $[a,b]$ iff $f(x)$ is continuous in $[a,b]$.

Corollary

A function f is of bounded variation on $[a,b]$ iff there is a bounded increasing function g on $[a,b]$ such that for any two points x' and x'' in $[a,b]$, $x' < x''$, we have

$$\left| f(x'') - f(x') \right| \le g(x'') - g(x')$$

Moreover, if g is continuous at x', so is f.

Proof

$$\text{Take } \quad g(x) = \begin{cases} V_a^x & , \ a < x \le b \\ 0 & , \ x = a \end{cases}$$

Then g is increasing and bounded on $[a,b]$.

Also, $\left| f(x') - f(x'') \right| \le V_{x'}^{x''}(f) = g(x'') - g(x')$

Which also yields that if g is continuous at x', so is f. ❑

Question

Show that the function f defined by

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & ; \ x \ne 0 \\ 0 & ; \ x = 0 \end{cases}$$

is of bounded variation on $[0,1]$.

Solution

f is differentiable on $[0,1]$ and $f'(x) = 2x\sin\dfrac{1}{x} - \sin x$ for $0 \le x \le 1$.

Also

$$\left| f'(x) \right| \le \left| 2x\sin\frac{1}{x} \right| + \left| \sin x \right| \le 2 + 1 = 3$$

i.e. $f'(x)$ is bounded on $[0,1]$

Hence f is of bounded variation on $[0,1]$. ❑

Question

Show that $g(x)=\begin{cases} x\cos\dfrac{\pi x}{2} & , \ 0<x\le 1 \\ 0 & , \ x=0 \end{cases}$ is not of bounded variation on $[0,1]$

Solution

Let $P=\left\{0,\dfrac{1}{2n},\dfrac{1}{2n-1},.....,\dfrac{1}{3},\dfrac{1}{2},1\right\}$ be a partition of $[0,1]$.

Then

$$\sum\left|f(x_i)-f(x_{i-1})\right|$$

$$=\left|f(1)-f\left(\frac{1}{2}\right)\right|+\left|f\left(\frac{1}{2}\right)-f\left(\frac{1}{3}\right)\right|+\left|f\left(\frac{1}{3}\right)-f\left(\frac{1}{4}\right)\right|+......+\left|f\left(\frac{1}{2n}\right)-f(0)\right|$$

$$=\left|\cos\frac{\pi}{2}-\frac{1}{2}\cos\frac{\pi}{4}\right|+\left|\frac{1}{2}\cos\frac{\pi}{4}-\frac{1}{3}\cos\frac{\pi}{6}\right|+\left|\frac{1}{3}\cos\frac{\pi}{6}-\frac{1}{4}\cos\frac{\pi}{8}\right|+.....+\left|\frac{1}{2n}\cos\frac{\pi}{4n}-0\right|$$

$$=2\left(\frac{1}{2}\cos\frac{\pi}{4}\right)+2\left(\frac{1}{3}\cos\frac{\pi}{6}\right)+2\left(\frac{1}{4}\cos\frac{\pi}{8}\right)+.....+2\left(\frac{1}{2n}\cos\frac{\pi}{4n}\right)$$

$$=2\left(\frac{1}{2}\cos\frac{\pi}{4}+\frac{1}{3}\cos\frac{\pi}{6}+\frac{1}{4}\cos\frac{\pi}{8}+......+\frac{1}{2n}\cos\frac{\pi}{4n}\right)$$

which is not bounded.

Hence $f(x)$ is not of bounded variation on $[0,1]$. ❑

Alternative

We have

$$\left|g(x_{k+1})-g(x_k)\right|+\left|g(x_k)-g(x_{k-1})\right|$$

$$=\left|\frac{1}{k+1}\cos\frac{(k+1)\pi}{2}-\frac{1}{k}\cos\frac{k\pi}{2}\right|+\left|\frac{1}{k}\cos\frac{k\pi}{2}-\frac{1}{k-1}\cos\frac{(k-1)\pi}{2}\right|$$

$$=\begin{cases} \dfrac{2}{k} & ; \ \text{if } k \text{ is even} \\ \dfrac{1}{k+1}+\dfrac{1}{k-1} & ; \ \text{if } k \text{ is odd} \end{cases}$$

$$\Rightarrow V_a^b(g) \le \sum_{k=1}^{n}\frac{1}{k} \le \sum_{k=1}^{\infty}\frac{1}{k}$$

$\because \displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ is divergent $\therefore V_a^b(g)$ is not finite.

Hence g is not of bounded variation. ❑

Chapter 8 – Improper Integrals.

We discussed Riemann-Stieltjes's integrals of the form $\int_a^b f\,d\alpha$ under the restrictions that both f and α are defined and bounded on a finite interval $[a,b]$. To extend the concept, we shall relax these restrictions on f and α.

➢ ***Definition***

The integral $\int_a^b f\,d\alpha$ is called an improper integral of first kind if $a=-\infty$ or $b=+\infty$ or *both* i.e. one or both integration limits is infinite.

➢ ***Definition***

The integral $\int_a^b f\,d\alpha$ is called an improper integral of second kind if $f(x)$ is unbounded at one or more points of $a\le x\le b$. Such points are called singularities of $f(x)$.

➢ ***Notations***

We shall denote the set of all functions f such that $f\in R(\alpha)$ on $[a,b]$ by $R(\alpha;a,b)$. When $\alpha(x)=x$, we shall simply write $R(a,b)$ for this set. The notation $\alpha\uparrow$ on $[a,\infty)$ will mean that α is monotonically increasing on $[a,\infty)$.

➢ ***Definition***

Assume that $f\in R(\alpha;a,b)$ for every $b\ge a$. Keep a,α and f fixed and define a function I on $[a,\infty)$ as follows:

$$I(b)=\int_a^b f(x)\,d\alpha(x) \quad \text{if} \quad b\ge a \;\ldots\ldots\ldots\ldots\; (i)$$

The function I so defined is called an infinite (or an improper) integral of first kind and is denoted by the symbol $\int_a^\infty f(x)\,d\alpha(x)$ or by $\int_a^\infty f\,d\alpha$.

The integral $\int_a^\infty f\,d\alpha$ is said to converge if the limit

$$\lim_{b\to\infty} I(b) \;\ldots\ldots\ldots\ldots\; (ii)$$

exists (finite). Otherwise, $\int_a^\infty f\,d\alpha$ is said to diverge.

If the limit in (ii) exists and equals A, the number A is called the value of the integral and we write $\int_a^\infty f\,d\alpha = A$

➢ ***Example***

Consider $\int_1^b x^{-p}\,dx$.

$\int_1^b x^{-p}\,dx = \dfrac{\left(1-b^{1-p}\right)}{p-1}$ if $p\ne 1$, the integral $\int_1^\infty x^{-p}\,dx$ diverges if $p<1$. When $p>1$, it converges and has the value $\dfrac{1}{p-1}$.

If $p=1$, we get $\int_1^b x^{-1}\,dx=\log b\to\infty$ as $b\to\infty$. $\Rightarrow \int_1^\infty x^{-1}\,dx$ diverges.

➢ *Example*

Consider $\int_0^b \sin 2\pi x\,dx$

$$\because \int_0^b \sin 2\pi x\,dx = \frac{(1-\cos 2\pi b)}{2\pi} \to \infty \quad \text{as} \quad b \to \infty .$$

$\therefore$ the integral $\int_0^\infty \sin 2\pi x\,dx$ diverges.

➢ *Note*

If $\int_{-\infty}^{a} f\,d\alpha$ and $\int_a^\infty f\,d\alpha$ are both convergent for some value of a, we say that the integral $\int_{-\infty}^{\infty} f\,d\alpha$ is convergent and its value is defined to be the sum

$$\int_{-\infty}^{\infty} f\,d\alpha = \int_{-\infty}^{a} f\,d\alpha + \int_a^{\infty} f\,d\alpha$$

The choice of the point a is clearly immaterial.

If the integral $\int_{-\infty}^{\infty} f\,d\alpha$ converges, its value is equal to the limit: $\lim_{b\to+\infty} \int_{-b}^{b} f\,d\alpha$.

➢ *Theorem*

Assume that $\alpha \uparrow$ on $[a,+\infty)$ and suppose that $f \in R(\alpha; a, b)$ for every $b \ge a$. Assume that $f(x) \ge 0$ for each $x \ge a$. Then $\int_a^\infty f\,d\alpha$ converges if, and only if, there exists a constant $M > 0$ such that

$$\int_a^b f\,d\alpha \le M \quad \text{for every} \quad b \ge a .$$

Proof

We have $I(b) = \int_a^b f(x)\,d\alpha(x), \quad b \ge a$

$\Rightarrow I \uparrow$ on $[a,+\infty)$

Then $\lim_{b\to+\infty} I(b) = \sup\{I(b) \mid b \ge a\} = M > 0$ and the theorem follows

$\Rightarrow \int_a^b f\,d\alpha \le M$ for every $b \ge a$ whenever the integral converges.

❁❁❁❁❁❁❁❁❁❁❁❁❁❁❁❁

➢ Theorem: (Comparison Test)

Assume that $\alpha \uparrow$ on $[a,+\infty)$. If $f \in R(\alpha; a,b)$ for every $b \ge a$, if $0 \le f(x) \le g(x)$ for every $x \ge a$, and if $\int_a^\infty g\,d\alpha$ converges, then $\int_a^\infty f\,d\alpha$ converges and we have

$$\int_a^\infty f\,d\alpha \le \int_a^\infty g\,d\alpha$$

Proof

Let $I_1(b) = \int_a^b f\,d\alpha$ and $I_2(b) = \int_a^b g\,d\alpha$, $b \ge a$

$\because$ $0 \le f(x) \le g(x)$ for every $x \ge a$

$\therefore$ $I_1(b) \le I_2(b)$ …………………. (i)

$\because$ $\int_a^\infty g\,d\alpha$ converges $\therefore$ $\exists$ a constant $M > 0$ such that

$$\int_a^\infty g\,d\alpha \le M \quad , \quad b \ge a \quad \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

From (i) and (ii) we have $I_1(b) \le M$, $b \ge a$.

$\Rightarrow$ $\lim_{b\to\infty} I_1(b)$ exists and is finite.

$\Rightarrow$ $\int_a^\infty f\,d\alpha$ converges.

Also $\lim_{b\to\infty} I_1(b) \le \lim_{b\to\infty} I_2(b) \le M$

$\Rightarrow$ $\int_a^\infty f\,d\alpha \le \int_a^\infty g\,d\alpha$.

➢ Theorem (Limit Comparison Test)

Assume that $\alpha \uparrow$ on $[a,+\infty)$. Suppose that $f \in R(\alpha; a,b)$ and that $g \in R(\alpha; a,b)$ for every $b \ge a$, where $f(x) \ge 0$ and $g(x) \ge 0$ if $x \ge a$. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

then $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ both converge or both diverge.

Proof

For all $b \ge a$, we can find some $N > 0$ such that

$$\left| \frac{f(x)}{g(x)} - 1 \right| < \varepsilon \qquad \forall \; x \ge N \quad \text{for every} \quad \varepsilon > 0.$$

$$\Rightarrow 1 - \varepsilon < \frac{f(x)}{g(x)} < 1 + \varepsilon$$

Let $\varepsilon = \frac{1}{2}$, then we have

$$\frac{1}{2} < \frac{f(x)}{g(x)} < \frac{3}{2}$$

$\Rightarrow$ $g(x) < 2f(x)$ ……….(i) and $2f(x) < 3g(x)$ ……….(ii)

From (i) $\int_a^{\infty} g\,d\alpha < 2\int_a^{\infty} f\,d\alpha$

$\Rightarrow \int_a^{\infty} g\,d\alpha$ converges if $\int_a^{\infty} f\,d\alpha$ converges and $\int_a^{\infty} f\,d\alpha$ diverges if $\int_a^{\infty} f\,d\alpha$ diverges.

From (ii) $2\int_a^{\infty} f\,d\alpha < 3\int_a^{\infty} g\,d\alpha$

$\Rightarrow \int_a^{\infty} f\,d\alpha$ converges if $\int_a^{\infty} g\,d\alpha$ converges and $\int_a^{\infty} g\,d\alpha$ diverges if $\int_a^{\infty} f\,d\alpha$ diverges.

$\Rightarrow$ The integrals $\int_a^{\infty} f\,d\alpha$ and $\int_a^{\infty} g\,d\alpha$ converge or diverge together.

➢ *Note*

The above theorem also holds if $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} = c$, provided that $c \neq 0$. If $c = 0$, we can only conclude that convergence of $\int_a^{\infty} g\,d\alpha$ implies convergence of $\int_a^{\infty} f\,d\alpha$.

➢ *Example*

For every real p, the integral $\int_1^{\infty} e^{-x}x^p\,dx$ converges.

This can be seen by comparison of this integral with $\int_1^{\infty} \dfrac{1}{x^2}\,dx$.

Since $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty}\dfrac{e^{-x}x^p}{1/x^2}$ where $f(x) = e^{-x}x^p$ and $g(x) = \dfrac{1}{x^2}$.

$$\Rightarrow \lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty} e^{-x}x^{p+2} = \lim_{x\to\infty}\frac{x^{p+2}}{e^x} = 0$$

and $\because \int_1^{\infty} \dfrac{1}{x^2}\,dx$ is convergent

$\therefore$ the given integral $\int_1^{\infty} e^{-x}x^p\,dx$ is also convergent.

➢ *Theorem*

Assume $\alpha \uparrow$ on $[a,+\infty)$. If $f \in R(\alpha; a, b)$ for every $b \geq a$ and if $\int_a^{\infty} |f|\,d\alpha$ converges, then $\int_a^{\infty} f\,d\alpha$ also converges.

Or: An absolutely convergent integral is convergent.

Proof

If $x \geq a$, $\pm f(x) \leq |f(x)|$

$\Rightarrow |f(x)| - f(x) \geq 0$

$\Rightarrow 0 \leq |f(x)| - f(x) \leq 2|f(x)|$

$$\Rightarrow \int_a^\infty (|f| - f)\, d\alpha \text{ converges.}$$

Subtracting from $\int_a^\infty |f|\, d\alpha$ we find that $\int_a^\infty f\, d\alpha$ converges.

($\because$ Difference of two convergent integrals is convergent)

➢ *Note*

$\int_a^\infty f\, d\alpha$ is said to converge absolutely if $\int_a^\infty |f|\, d\alpha$ converges. It is said to be convergent conditionally if $\int_a^\infty f\, d\alpha$ converges but $\int_a^\infty |f|\, d\alpha$ diverges.

➢ *Remark*

Every absolutely convergent integral is convergent.

➢ *Theorem*

Let f be a positive decreasing function defined on $[a,+\infty)$ such that $f(x) \to 0$ as $x \to +\infty$. Let α be bounded on $[a,+\infty)$ and assume that $f \in R(\alpha; a, b)$ for every $b \geq a$. Then the integral $\int_a^\infty f\, d\alpha$ is convergent.

Proof

Integration by parts gives

$$\int_a^b f\, d\alpha = \big| f(x)\cdot\alpha(x) \big|_a^b - \int_a^b \alpha(x)\, df$$

$$= f(b)\cdot\alpha(b) - f(a)\cdot\alpha(a) + \int_a^b \alpha\ d(-f)$$

It is obvious that $f(b)\alpha(b) \to 0$ as $b \to +\infty$

($\because$ α is bounded and $f(x) \to 0$ as $x \to +\infty$)

and $f(a)\alpha(a)$ is finite.

$\therefore$ the convergence of $\int_a^b f\, d\alpha$ depends upon the convergence of $\int_a^b \alpha\ d(-f)$.

Actually, this integral converges absolutely. To see this, suppose $|\alpha(x)| \leq M$ for all $x \geq a$ ($\because$ $\alpha(x)$ is given to be bounded)

$$\Rightarrow \int_a^b |\alpha(x)|\, d(-f) \leq \int_a^b M\, d(-f)$$

But $\int_a^b M\, d(-f) = M \big| -f \big|_a^b = M\ f(a) - M\ f(b) \to M\ f(a)$ as $b \to \infty$.

$$\Rightarrow \int_a^\infty M\, d(-f) \text{ is convergent.}$$

$\because$ $-f$ is an increasing function.

$$\therefore \int_a^\infty |\alpha|\, d(-f) \text{ is convergent.} \quad \text{(Comparison Test)}$$

$$\Rightarrow \int_a^\infty f\, d\alpha \text{ is convergent.}$$

❁❁❁❁❁❁❁❁❁❁❁❁❁❁❁❁

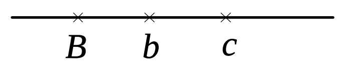
➢ ***Theorem (Cauchy condition for infinite integrals)***

Assume that $f \in R(\alpha; a, b)$ for every $b \geq a$. Then the integral $\int_a^\infty f\,d\alpha$ converges if, and only if, for every $\varepsilon > 0$ there exists a $B > 0$ such that $c > b > B$ implies

$$\left| \int_b^c f(x)\,d\alpha(x) \right| < \varepsilon$$

Proof

Let $\int_a^\infty f\,d\alpha$ be convergent. Then $\exists \; B > 0$ such that



$$\left| \int_a^b f\,d\alpha - \int_a^\infty f\,d\alpha \right| < \frac{\varepsilon}{2} \quad \text{for every } \; b \geq B \; \ldots\ldots\ldots..(i)$$

Also for $c > b > B$,

$$\left| \int_a^c f\,d\alpha - \int_a^\infty f\,d\alpha \right| < \frac{\varepsilon}{2} \; \ldots\ldots\ldots\ldots\ldots.. \; (ii)$$

Consider

$$\left| \int_b^c f\,d\alpha \right| = \left| \int_a^c f\,d\alpha - \int_a^b f\,d\alpha \right|$$

$$= \left| \int_a^c f\,d\alpha - \int_a^\infty f\,d\alpha + \int_a^\infty f\,d\alpha - \int_a^b f\,d\alpha \right|$$

$$\leq \left| \int_a^c f\,d\alpha - \int_a^\infty f\,d\alpha \right| + \left| \int_a^\infty f\,d\alpha - \int_a^b f\,d\alpha \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \left| \int_b^c f\,d\alpha \right| < \varepsilon \quad \text{when } \; c > b > B.$$

Conversely, assume that the Cauchy condition holds.

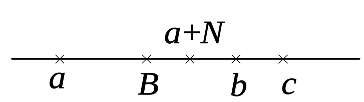
Define $a_n = \int_a^{a+n} f\,d\alpha$ if $n = 1, 2, \ldots\ldots$

The sequence $\{a_n\}$ is a Cauchy sequence $\Rightarrow$ it converges.

Let $\lim_{n\to\infty} a_n = A$

Given $\varepsilon > 0$, choose B so that $\left| \int_b^c f\,d\alpha \right| < \frac{\varepsilon}{2}$ if $c > b > B$.

and also that $|a_n - A| < \frac{\varepsilon}{2}$ whenever $a + n \geq B$.



Choose an integer N such that $a + N > B$ i.e. $N > B - a$

Then, if $b > a + N$, we have

$$\left| \int_a^b f\,d\alpha - A \right| = \left| \int_a^{a+N} f\,d\alpha - A + \int_{a+N}^b f\,d\alpha \right|$$

$$\leq |a_N - A| + \left| \int_{a+N}^b f\,d\alpha \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \int_a^\infty f\,d\alpha = A$$

This completes the proof.

➢ Remarks

It follows from the above theorem that convergence of $\int_a^{\infty} f\,d\alpha$ implies $\lim_{b\to\infty}\int_b^{b+\varepsilon} f\,d\alpha = 0$ for every fixed $\varepsilon > 0$.

However, this does not imply that $f(x)\to 0$ as $x\to\infty$.

➢ Theorem

Every convergent infinite integral $\int_a^{\infty} f(x)\,d\alpha(x)$ can be written as a convergent infinite series. In fact, we have

$$\int_a^{\infty} f(x)\,d\alpha(x) = \sum_{k=1}^{\infty} a_k \quad \text{where } a_k = \int_{a+k-1}^{a+k} f(x)\,d\alpha(x) \quad \text{……….. (1)}$$

Proof

$\because \int_a^{\infty} f\,d\alpha$ converges, the sequence $\left\{\int_a^{a+n} f\,d\alpha\right\}$ also converges.

But $\int_a^{a+n} f\,d\alpha = \sum_{k=1}^{n} a_k$. Hence the series $\sum_{k=1}^{\infty} a_k$ converges and equals $\int_a^{\infty} f\,d\alpha$.

➢ Remarks

It is to be noted that the convergence of the series in (1) does not always imply convergence of the integral. For example, suppose $a_k = \int_{k-1}^{k} \sin 2\pi x\,dx$. Then each $a_k = 0$ and $\sum a_k$ converges.

However, the integral $\int_0^{\infty} \sin 2\pi x\,dx = \lim_{b\to\infty}\int_0^{b} \sin 2\pi x\,dx = \lim_{b\to\infty}\dfrac{1-\cos 2\pi b}{2\pi}$ diverges.

IMPROPER INTEGRAL OF THE SECOND KIND

➢ Definition

Let f be defined on the half open interval $(a,b]$ and assume that $f \in R(\alpha; x, b)$ for every $x \in (a,b]$. Define a function I on $(a,b]$ as follows:

$$I(x) = \int_x^{b} f\,d\alpha \quad \text{if } x \in (a,b] \quad \text{……….. }(i)$$

The function I so defined is called an improper integral of the second kind and is denoted by the symbol $\int_{a+}^{b} f(t)\,d\alpha(t)$ or $\int_{a+}^{b} f\,d\alpha$.

The integral $\int_{a+}^{b} f\,d\alpha$ is said to converge if the limit

$$\lim_{x\to a+} I(x) \quad \text{……...}(ii) \quad \text{exists (finite).}$$

Otherwise, $\int_{a+}^{b} f\,d\alpha$ is said to diverge. If the limit in (ii) exists and equals A, the number A is called the value of the integral and we write $\int_{a+}^{b} f\,d\alpha = A$.

Similarly, if f is defined on $[a,b)$ and $f \in R(\alpha; a, x)$ $\forall\ x \in [a,b)$ then $I(x) = \int_a^x f\, d\alpha$ if $x \in [a,b)$ is also an improper integral of the second kind and is denoted as $\int_a^{b-} f\, d\alpha$ and is convergent if $\lim_{x \to b-} I(x)$ exists (finite).

➢ ***Example***

$f(x) = x^{-p}$ is defined on $(0,b]$ and $f \in R(x,b)$ for every $x \in (0,b]$.

$$I(x) = \int_x^b x^{-p}\, dx \qquad \text{if} \quad x \in (0,b]$$

$$= \int_{0+}^b x^{-p}\, dx = \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^b x^{-p}\, dx$$

$$= \lim_{\varepsilon \to 0} \left| \frac{x^{1-p}}{1-p} \right|_{\varepsilon}^{b} = \lim_{\varepsilon \to 0} \frac{b^{1-p} - \varepsilon^{1-p}}{1-p} \qquad , \quad (p \neq 1)$$

$$= \begin{cases} \textit{finite} \ , & p < 1 \\ \textit{infinite} \ , & p > 1 \end{cases}$$

When $p = 1$, we get $\int_{\varepsilon}^{b} \frac{1}{x}\, dx = \log b - \log \varepsilon \to \infty$ as $\varepsilon \to 0$.

$\Rightarrow \int_{0+}^{b} x^{-1}\, dx$ also diverges.

Hence the integral converges when $p < 1$ and diverges when $p \geq 1$.

➢ ***Note***

If the two integrals $\int_{a+}^{c} f\, d\alpha$ and $\int_{c}^{b-} f\, d\alpha$ both converge, we write

$$\int_{a+}^{b-} f\, d\alpha = \int_{a+}^{c} f\, d\alpha + \int_{c}^{b-} f\, d\alpha$$

The definition can be extended to cover the case of any finite number of sums. We can also consider mixed combinations such as

$$\int_{a+}^{b} f\, d\alpha + \int_{b}^{\infty} f\, d\alpha \quad \text{which can be written as} \quad \int_{a+}^{\infty} f\, d\alpha .$$

➢ ***Example***

Consider $\int_{0+}^{\infty} e^{-x} x^{p-1}\, dx$, $(p > 0)$

This integral must be interpreted as a sum as

$$\int_{0+}^{\infty} e^{-x} x^{p-1}\, dx = \int_{0+}^{1} e^{-x} x^{p-1}\, dx + \int_{1}^{\infty} e^{-x} x^{p-1}\, dx$$

$$= I_1 + I_2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (i)$$

I_2, the second integral, converges for every real p as proved earlier.

To test I_1, put $t = \frac{1}{x} \quad \Rightarrow \ dx = -\frac{1}{t^2}\, dt$

$$\Rightarrow\ I_1=\lim_{\varepsilon\to0}\int_{\varepsilon}^{1}e^{-x}x^{p-1}\,dx\ =\lim_{\varepsilon\to0}\int_{1/\varepsilon}^{1}e^{-\frac{1}{t}}t^{1-p}\left(-\frac{1}{t^2}dt\right)\ =\lim_{\varepsilon\to0}\int_{1}^{1/\varepsilon}e^{-\frac{1}{t}}t^{-p-1}\,dt$$

Take $f(t)=e^{-\frac{1}{t}}\,t^{-p-1}$ and $g(t)=t^{-p-1}$

Then $\displaystyle\lim_{t\to\infty}\frac{f(t)}{g(t)}=\lim_{t\to\infty}\frac{e^{-\frac{1}{t}}\cdot t^{-p-1}}{t^{-p-1}}=1$ and since $\displaystyle\int_{1}^{\infty}t^{-p-1}\,dt$ converges when $p>0$

$\therefore\ \displaystyle\int_{1}^{\infty}e^{-\frac{1}{t}}t^{-p-1}\,dt$ converges when $p>0$

Thus $\displaystyle\int_{0+}^{\infty}e^{-x}\,x^{p-1}\,dx$ converges when $p>0$.

When $p>0$, the value of the sum in (*i*) is denoted by $\Gamma(p)$. The function so defined is called the Gamma function.

➢ *Note*

The tests developed to check the behaviour of the improper integrals of Ist kind are applicable to improper integrals of IInd kind after making necessary modifications.

➢ *A Useful Comparison Integral*

$$\int_{a}^{b}\frac{dx}{(x-a)^n}$$

We have, if $n\neq1$,

$$\int_{a+\varepsilon}^{b}\frac{dx}{(x-a)^n}=\left|\frac{1}{(1-n)(x-a)^{n-1}}\right|_{a+\varepsilon}^{b}$$

$$=\frac{1}{(1-n)}\left(\frac{1}{(b-a)^{n-1}}-\frac{1}{\varepsilon^{n-1}}\right)$$

Which tends to $\dfrac{1}{(1-n)(b-a)^{n-1}}$ or $+\infty$ according as $n<1$ or $n>1$, as $\varepsilon\to0$.

Again, if $n=1$,

$$\int_{a+\varepsilon}^{b}\frac{dx}{x-a}=\log(b-a)-\log\varepsilon\to+\infty\quad\text{as}\quad\varepsilon\to0.$$

Hence the improper integral $\displaystyle\int_{a}^{b}\frac{dx}{(x-a)^n}$ converges iff $n<1$.

❁❁❁❁❁❁❁❁❁❁❁❁❁❁❁❁

➢ *Question*

Examine the convergence of

(*i*) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}\left(1+x^2\right)}$ (*ii*) $\displaystyle\int_0^1 \frac{dx}{x^2(1+x)^2}$ (*iii*) $\displaystyle\int_0^1 \frac{dx}{x^{1/2}\left(1-x\right)^{1/3}}$

Solution

(*i*) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}\left(1+x^2\right)}$

Here '0' is the only point of infinite discontinuity of the integrand.
We have

$$f(x)=\frac{1}{x^{1/3}\left(1+x^2\right)}$$

Take $g(x)=\dfrac{1}{x^{1/3}}$

Then $\displaystyle\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1}{1+x^2}=1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ have identical behaviours.

$\because \displaystyle\int_0^1 \frac{dx}{x^{1/3}}$ converges $\therefore \displaystyle\int_0^1 \frac{dx}{x^{1/3}\left(1+x^2\right)}$ also converges.

(*ii*) $\displaystyle\int_0^1 \frac{dx}{x^2(1+x)^2}$

Here '0' is the only point of infinite discontinuity of the given integrand.
We have

$$f(x)=\frac{1}{x^2(1+x)^2}$$

Take $g(x)=\dfrac{1}{x^2}$

Then $\displaystyle\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1}{(1+x)^2}=1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ behave alike.

But $n=2$ being greater than 1, the integral $\int_0^1 g(x)\,dx$ does not converge. Hence the given integral also does not converge.

(*iii*) $\displaystyle\int_0^1 \frac{dx}{x^{1/2}\left(1-x\right)^{1/3}}$

Here '0' and '1' are the two points of infinite discontinuity of the integrand.
We have

$$f(x)=\frac{1}{x^{1/2}\left(1-x\right)^{1/3}}$$

We take any number between 0 and 1, say $1/2$, and examine the convergence of the improper integrals $\int_0^{1/2} f(x)\,dx$ and $\int_{1/2}^1 f(x)\,dx$.

To examine the convergence of $\int_0^{1/2} \frac{1}{x^{1/2}(1-x)^{1/3}}dx$, we take $g(x)=\frac{1}{x^{1/2}}$

Then

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1}{(1-x)^{1/3}}=1$$

$\because \int_0^{1/2}\frac{1}{x^{1/2}}dx$ converges $\therefore \int_0^{1/2}\frac{1}{x^{1/2}(1-x)^{1/3}}dx$ is convergent.

To examine the convergence of $\int_{1/2}^1 \frac{1}{x^{1/2}(1-x)^{1/3}}dx$, we take $g(x)=\frac{1}{(1-x)^{1/3}}$

Then

$$\lim_{x\to 1}\frac{f(x)}{g(x)}=\lim_{x\to 1}\frac{1}{x^{1/2}}=1$$

$\because \int_{1/2}^1\frac{1}{(1-x)^{1/3}}dx$ converges $\because \int_{1/2}^1\frac{1}{x^{1/2}(1-x)^{1/3}}dx$ is convergent.

Hence $\int_0^1 f(x)\,dx$ converges.

➢ *Question*

Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ exists iff m, n are both positive.

Solution

The integral is proper if $m\geq 1$ and $n\geq 1$.

The number ‘0’ is a point of infinite discontinuity if $m<1$ and the number ‘1’ is a point of infinite discontinuity if $n<1$.

Let $m<1$ and $n<1$.

We take any number, say $1/2$, between 0 & 1 and examine the convergence of the improper integrals $\int_0^{1/2} x^{m-1}(1-x)^{n-1}dx$ and $\int_{1/2}^1 x^{m-1}(1-x)^{n-1}dx$ at ‘0’ and ‘1’ respectively.

Convergence at 0:

We write

$$f(x)=x^{m-1}(1-x)^{n-1}=\frac{(1-x)^{n-1}}{x^{1-m}} \quad \text{and take} \quad g(x)=\frac{1}{x^{1-m}}$$

Then $\frac{f(x)}{g(x)}\to 1$ as $x\to 0$

As $\int_0^{1/2}\frac{1}{x^{1-m}}dx$ is convergent at 0 iff $1-m<1$ i.e. $m>0$

We deduce that the integral $\int_0^{1/2} x^{m-1}(1-x)^{n-1}dx$ is convergent at 0, iff m is +ive.

Convergence at 1:

We write $f(x)=x^{m-1}(1-x)^{n-1}=\dfrac{x^{m-1}}{(1-x)^{1-n}}$ and take $g(x)=\dfrac{1}{(1-x)^{1-n}}$

Then $\dfrac{f(x)}{g(x)}\to 1$ as $x\to 1$

As $\displaystyle\int_{1/2}^{1}\frac{1}{(1-x)^{1-n}}dx$ is convergent, iff $1-n<1$ i.e. $n>0$.

We deduce that the integral $\displaystyle\int_{1/2}^{1}x^{m-1}\left(1-x\right)^{n-1}dx$ converges iff $n>0$.

Thus $\displaystyle\int_{0}^{1}x^{m-1}\left(1-x\right)^{n-1}dx$ exists for positive values of m,n only.

It is a function which depends upon m & n and is defined for all positive values of m & n. It is called Beta function.

➢ ***Question***

Show that the following improper integrals are convergent.

(i) $\displaystyle\int_{1}^{\infty}\sin^2\frac{1}{x}\,dx$ (ii) $\displaystyle\int_{1}^{\infty}\frac{\sin^2 x}{x^2}\,dx$ (iii) $\displaystyle\int_{0}^{1}\frac{x\log x}{(1+x)^2}\,dx$ (iv) $\displaystyle\int_{0}^{1}\log x\cdot\log(1+x)\,dx$

Solution

(*i*) Let $f(x)=\sin^2\dfrac{1}{x}$ and $g(x)=\dfrac{1}{x^2}$

then $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\sin^2\frac{1}{x}}{\frac{1}{x^2}}=\lim_{y\to 0}\left(\frac{\sin y}{y}\right)^2=1$

$\Rightarrow$ $\displaystyle\int_{1}^{\infty}f(x)\,dx$ and $\displaystyle\int_{1}^{\infty}\frac{1}{x^2}\,dx$ behave alike.

$\because$ $\displaystyle\int_{1}^{\infty}\frac{1}{x^2}\,dx$ is convergent $\therefore$ $\displaystyle\int_{1}^{\infty}\sin^2\frac{1}{x}\,dx$ is also convergent.

(*ii*) $\displaystyle\int_{1}^{\infty}\frac{\sin^2 x}{x^2}\,dx$

Take $f(x)=\dfrac{\sin^2 x}{x^2}$ and $g(x)=\dfrac{1}{x^2}$

$\sin^2 x\le 1$ $\Rightarrow$ $\dfrac{\sin^2 x}{x^2}\le\dfrac{1}{x^2}$ $\forall$ $x\in\left(1,\infty\right)$

and $\displaystyle\int_{1}^{\infty}\frac{1}{x^2}\,dx$ converges $\therefore$ $\displaystyle\int_{1}^{\infty}\frac{\sin^2 x}{x^2}\,dx$ converges.

➢ ***Note***

$\displaystyle\int_{0}^{1}\frac{\sin^2 x}{x^2}\,dx$ is a proper integral because $\displaystyle\lim_{x\to 0}\frac{\sin^2 x}{x^2}=1$ so that '0' is not a point of infinite discontinuity. Therefore $\displaystyle\int_{0}^{\infty}\frac{\sin^2 x}{x^2}\,dx$ is convergent.

(iii) $\displaystyle\int_0^1 \frac{x\log x}{(1+x)^2}\,dx$

$\because\ \log x < x\ ,\quad x\in(0,1)$

$\therefore\ x\log x < x^2$

$$\Rightarrow\ \frac{x\log x}{(1+x)^2} < \frac{x^2}{(1+x)^2}$$

Now $\displaystyle\int_0^1 \frac{x^2}{(1+x)^2}\,dx$ is a proper integral.

$\therefore\ \displaystyle\int_0^1 \frac{x\log x}{(1+x)^2}\,dx$ is convergent.

(iv) $\displaystyle\int_0^1 \log x\cdot\log(1+x)\,dx$

$\because\ \log x < x \qquad \therefore\ \log(x+1) < x+1$

$\Rightarrow\ \log x\cdot\log(1+x) < x(x+1)$

$\because\ \displaystyle\int_0^1 x(x+1)\,dx$ is a proper integral $\quad\therefore\ \displaystyle\int_0^1 \log x\cdot\log(1+x)\,dx$ is convergent.

➢ ***Note***

(i) $\displaystyle\int_0^a \frac{1}{x^p}\,dx$ diverges when $p\ge 1$ and converges when $p<1$.

(ii) $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ converges iff $p>1$.

UNIFORM CONVERGENCE OF IMPROPER INTEGRALS

➢ ***Definition***

Let f be a real valued function of two variables x & y, $x\in[a,+\infty)$, $y\in S$ where $S\subset\mathbb{R}$. Suppose further that, for each y in S, the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ is convergent. If F denotes the function defined by the equation

$$F(y)=\int_a^\infty f(x,y)\,d\alpha(x) \qquad \text{if} \quad y\in S$$

the integral is said to converge *pointwise* to F on S

➢ ***Definiton***

Assume that the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges pointwise to F on S. The integral is said to converge Uniformly on S if, for every $\varepsilon>0$ there exists a $B>0$ (depending only on ε) such that $b>B$ implies

$$\left|F(y)-\int_a^b f(x,y)\,d\alpha(x)\right| < \varepsilon \qquad \forall\ y\in S.$$

(Pointwise convergence means convergence when y is fixed but uniform convergence is for every $y\in S$).

> **Theorem (Cauchy condition for uniform convergence.)**
>
> The integral $\int_a^{\infty} f(x,y)\,d\alpha(x)$ converges uniformly on S, iff, for every $\varepsilon > 0$ there exists a $B > 0$ (depending on ε) such that $c > b > B$ implies
>
> $$\left| \int_b^c f(x,y)\,d\alpha(x) \right| < \varepsilon \qquad \forall \;\; y \in S .$$

Proof

Proceed as in the proof for Cauchy condition for infinite integral $\int_a^{\infty} f\,d\alpha$.

> **Theorem (Weierstrass M-test)**
>
> Assume that $\alpha \uparrow$ on $[a,+\infty)$ and suppose that the integral $\int_a^b f(x,y)\,d\alpha(x)$ exists for every $b \geq a$ and for every y in S. If there is a positive function M defined on $[a,+\infty)$ such that the integral $\int_a^{\infty} M(x)\,d\alpha(x)$ converges and $|f(x,y)| \leq M(x)$ for each $x \geq a$ and every y in S, then the integral $\int_a^{\infty} f(x,y)\,d\alpha(x)$ converges uniformly on S.

Proof

$\because$ $|f(x,y)| \leq M(x)$ for each $x \geq a$ and every y in S.

$\therefore$ For every $c \geq b$, we have

$$\left| \int_b^c f(x,y)\,d\alpha(x) \right| \leq \int_b^c |f(x,y)\,d\alpha(x)| \leq \int_b^c M\,d\alpha \;\; \ldots\ldots\ldots\ldots \; (i)$$

$\because$ $I = \int_a^{\infty} M\,d\alpha$ is convergent

$\therefore$ given $\varepsilon > 0$, $\exists$ $B > 0$ such that $b > B$ implies

$$\left| \int_a^b M\,d\alpha - I \right| < \varepsilon/2 \;\; \ldots\ldots\ldots\ldots\ldots \; (ii)$$

Also if $c > b > B$, then

$$\left| \int_a^c M\,d\alpha - I \right| < \varepsilon/2 \;\; \ldots\ldots\ldots\ldots\ldots \; (iii)$$

Then $\left| \int_b^c M\,d\alpha \right| = \left| \int_a^c M\,d\alpha - \int_a^b M\,d\alpha \right|$

$$= \left| \int_a^c M\,d\alpha - I + I - \int_a^b M\,d\alpha \right|$$

$$\leq \left| \int_a^c M\,d\alpha - I \right| + \left| \int_a^b M\,d\alpha - I \right| < \varepsilon/2 + \varepsilon/2 = \varepsilon \qquad \text{(By } ii \text{ \& } iii\text{)}$$

$$\Rightarrow \left| \int_b^c f(x,y)\,d\alpha(x) \right| < \varepsilon , \quad c > b > B \text{ \& for each } y \in S$$

Cauchy condition for convergence (uniform) being satisfied.

Therefore the integral $\int_a^{\infty} f(x,y)\,d\alpha(x)$ converges uniformly on S.

❁❁❁❁❁❁❁❁❁❁❁❁❁❁❁❁

➢ *Example*

Consider $\int_0^\infty e^{-xy}\sin x\,dx$

$$\left|e^{-xy}\sin x\right| \le \left|e^{-xy}\right| = e^{-xy} \qquad \left(\because \left|\sin x\right|\le 1\right)$$

and $\qquad e^{-xy} \le e^{-xc} \qquad$ if $\quad c\le y$

Now take $\quad M(x)=e^{-cx}$

The integral $\int_0^\infty M(x)\,dx=\int_0^\infty e^{-cx}\,dx$ is convergent & converging to $\frac{1}{c}$.

$\therefore$ The conditions of M-test are satisfied and $\int_0^\infty e^{-xy}\sin x\,dx$ converges uniformly on $[c,+\infty)$ for every $c>0$.

➢ *Theorem (Dirichlet's test for uniform convergence)*

Assume that α is bounded on $[a,+\infty)$ and suppose the integral $\int_a^b f(x,y)\,d\alpha(x)$ exists for every $b\ge a$ and for every y in S. For each fixed y in S, assume that $f(x,y)\le f(x',y)$ if $a\le x'<x<+\infty$. Furthermore, suppose there exists a positive function g, defined on $[a,+\infty)$, such that $g(x)\to 0$ as $x\to+\infty$ and such that $x\ge a$ implies

$$\left|f(x,y)\right| \le g(x) \quad \text{for every } y \text{ in } S.$$

Then the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S.

Proof

Let $M>0$ be an upper bound for $|\alpha|$ on $[a,+\infty)$.

Given $\varepsilon>0$, choose $B>a$ such that $x\ge B$ implies

$$g(x)<\frac{\varepsilon}{4M}$$

($\because$ $g(x)$ is +ive and $\to 0$ as $x\to\infty$ $\therefore$ $\left|g(x)-0\right|<\frac{\varepsilon}{4M}$ for $x\ge B$)

If $c>b$, integration by parts yields

$$\int_b^c f\,d\alpha = \left| f(x,y)\cdot\alpha(x)\right|_b^c - \int_b^c \alpha\,df$$

$$= f(c,y)\alpha(c)-f(b,y)\alpha(b)+\int_b^c \alpha\,d(-f) \;\ldots\ldots\ldots\ldots\; (i)$$

But, since $-f$ is increasing (for each fixed y), we have

$$\left|\int_b^c \alpha\,d(-f)\right| \le M\int_b^c d(-f) \qquad (\because \text{ upper bound of } |\alpha| \text{ is } M)$$

$$= M\,f(b,y)-M\,f(c,y) \;\ldots\ldots\ldots\ldots\ldots\; (ii)$$

Now if $c>b>B$, we have from (i) and (ii)

$$\left|\int_b^c f\,d\alpha\right| \le \left| f(c,y)\alpha(c)-f(b,y)\alpha(b)\right| + \left|\int_b^c \alpha\,d(-f)\right|$$

$$\le \left|\alpha(c)\right|\left|f(c,y)\right|+\left|f(b,y)\right|\left|\alpha(b)\right|+M\left|f(b,y)-f(c,y)\right|$$

$$\le \left|\alpha(c)\right|\left|f(c,y)\right|+\left|\alpha(b)\right|\left|f(b,y)\right|+M\left|f(b,y)\right|+M\left|f(c,y)\right|$$

$$\leq M\,g(c) + M\,g(b) + M\,g(b) + M\,g(c)$$
$$= 2M\left[g(b) + g(c)\right]$$
$$< 2M\left[\frac{\varepsilon}{4M} + \frac{\varepsilon}{4M}\right] = \varepsilon$$

$$\Rightarrow \left|\int_b^c f\,d\alpha\right| < \varepsilon \quad \text{for every } y \text{ in } S.$$

Therefore the Cauchy condition is satisfied and $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S.

➢ *Example*

Consider $\int_0^\infty \frac{e^{-xy}}{x}\sin x\,dx$

Take $\alpha(x) = \cos x$ and $f(x,y) = \frac{e^{-xy}}{x}$ if $x > 0,\ y \geq 0$.

If $S = [0,+\infty)$ and $g(x) = \frac{1}{x}$ on $[\varepsilon,+\infty)$ for every $\varepsilon > 0$ then

i) $f(x,y) \leq f(x',y)$ if $x' \leq x$ and $\alpha(x)$ is bounded on $[\varepsilon,+\infty)$.

ii) $g(x) \to 0$ as $x \to +\infty$

iii) $|f(x,y)| = \left|\frac{e^{-xy}}{x}\right| \leq \frac{1}{x} = g(x) \quad \forall\ y \in S$.

So that the conditions of Dirichlet's theorem are satisfied.
Hence

$$\int_\varepsilon^\infty \frac{e^{-xy}}{x}\sin x\,dx = +\int_\varepsilon^\infty \frac{e^{-xy}}{x}\,d(-\cos x) \text{ converges uniformly on } [\varepsilon,+\infty) \text{ if } \varepsilon > 0.$$

$\because \lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \therefore \int_0^\varepsilon e^{-xy}\frac{\sin x}{x}dx$ converges being a proper integral.

$\Rightarrow \int_0^\infty e^{-xy}\frac{\sin x}{x}dx$ also converges uniformly on $[0,+\infty)$.

➢ *Remarks*

Dirichlet's test can be applied to test the convergence of the integral of a product. For this purpose the test can be modified and restated as follows:

Let $\phi(x)$ be bounded and monotonic in $[a,+\infty)$ and let $\phi(x) \to 0$, when $x \to \infty$. Also let $\int_a^X f(x)\,dx$ be bounded when $X \geq a$.

Then $\int_a^\infty f(x)\phi(x)\,dx$ is convergent.

➢ *Example*

Consider $\int_0^\infty \frac{\sin x}{x}dx$

$\because \frac{\sin x}{x} \to 1$ as $x \to 0$.

$\therefore$ 0 is not a point of infinite discontinuity.

Now consider the improper integral $\int_1^{\infty} \frac{\sin x}{x} dx$.

The factor $\frac{1}{x}$ of the integrand is monotonic and $\to 0$ as $x \to \infty$.

Also $\left| \int_1^X \sin x \, dx \right| = \left| -\cos X + \cos(1) \right| \le \left| \cos X \right| + \left| \cos(1) \right| < 2$

So that $\int_1^X \sin x \, dx$ is bounded above for every $X \ge 1$.

$\Rightarrow \int_1^{\infty} \frac{\sin x}{x} dx$ is convergent. Now since $\int_0^1 \frac{\sin x}{x} dx$ is a proper integral, we see that $\int_0^{\infty} \frac{\sin x}{x} dx$ is convergent.

➢ ***Example***

Consider $\int_0^{\infty} \sin x^2 \, dx$.

We write $\sin x^2 = \frac{1}{2x} \cdot 2x \cdot \sin x^2$

Now $\int_1^{\infty} \sin x^2 \, dx = \int_1^{\infty} \frac{1}{2x} \cdot 2x \cdot \sin x^2 \, dx$

$\frac{1}{2x}$ is monotonic and $\to 0$ as $x \to \infty$.

Also $\left| \int_1^X 2x \sin x^2 \, dx \right| = \left| -\cos X^2 + \cos(1) \right| < 2$

So that $\int_1^X 2x \sin x^2 \, dx$ is bounded for $X \ge 1$.

Hence $\int_1^{\infty} \frac{1}{2x} \cdot 2x \cdot \sin x^2 \, dx$ i.e. $\int_1^{\infty} \sin x^2 \, dx$ is convergent.

Since $\int_0^1 \sin x^2 \, dx$ is only a proper integral, we see that the given integral is convergent.

➢ ***Example***

Consider $\int_0^{\infty} e^{-ax} \frac{\sin x}{x} dx$, $a > 0$

Here e^{-ax} is monotonic and bounded and $\int_0^{\infty} \frac{\sin x}{x} dx$ is convergent.

Hence $\int_0^{\infty} e^{-ax} \frac{\sin x}{x} dx$ is convergent.

➢ Example

Show that $\int_0^\infty \frac{\sin x}{x}dx$ is not absolutely convergent.

Solution

Consider the proper integral $\int_0^{n\pi} \frac{|\sin x|}{x}dx$ where n is a positive integer. We have

We need not take $|x|$ because $x \ge 0$.

$$\int_0^{n\pi} \frac{|\sin x|}{x}dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}dx$$

Put $x = (r-1)\pi + y$ so that y varies in $[0,\pi]$.

We have $|\sin[(r-1)\pi + y]| = |(-1)^{r-1}\sin y| = \sin y$

$$\therefore \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}dx = \int_0^{\pi} \frac{\sin y}{(r-1)\pi + y}dy$$

$\because$ $r\pi$ is the max. value of $[(r-1)\pi + y]$ in $[0,\pi]$

$$\therefore \int_0^{\pi} \frac{\sin y}{(r-1)\pi + y}dy \ge \frac{1}{r\pi}\int_0^{\pi} \sin y\, dy = \frac{2}{r\pi}$$

$\left[\because\right.$ Division by max. value will lessen the value

$$\Rightarrow \int_0^{n\pi} \frac{|\sin x|}{x}dx \ge \sum_1^n \frac{2}{r\pi} = \frac{2}{\pi}\sum_1^n \frac{1}{r}$$

$\because$ $\sum_1^n \frac{1}{r} \to \infty$ as $n \to \infty$, we see that

$$\int_0^{n\pi} \frac{|\sin x|}{x}dx \to \infty \quad \text{as} \quad n \to \infty.$$

Let, now, X be any real number.
There exists a +tive integer n such that $n\pi \le X < (n+1)\pi$.

We have $\int_0^{X} \frac{|\sin x|}{x}dx \ge \int_0^{n\pi} \frac{|\sin x|}{x}dx$

Let $X \to \infty$ so that n also $\to \infty$. Then we see that $\int_0^{X} \frac{|\sin x|}{x}dx \to \infty$

So that $\int_0^{\infty} \frac{|\sin x|}{x}dx$ does not converge.

➢ Questions

Examine the convergence of

(*i*) $\int_1^\infty \frac{x}{(1+x)^3}dx$ (*ii*) $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}dx$ (*iii*) $\int_1^\infty \frac{dx}{x^{1/3}(1+x)^{1/2}}$

Solution

(***i***) Let $f(x) = \frac{x}{(1+x)^3}$ and take $g(x) = \frac{x}{x^3} = \frac{1}{x^2}$

As $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^3}{(1+x)^3} = 1$

Therefore the two integrals $\int_1^\infty \frac{x}{(1+x)^3}dx$ and $\int_1^\infty \frac{1}{x^2}dx$ have identical behaviour for convergence at ∞.

$\because \int_1^\infty \frac{1}{x^2}dx$ is convergent $\therefore \int_1^\infty \frac{x}{(1+x)^3}dx$ is convergent.

(***ii***) Let $f(x)=\frac{1}{(1+x)\sqrt{x}}$ and take $g(x)=\frac{1}{x\sqrt{x}}=\frac{1}{x^{3/2}}$

We have $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=\lim\limits_{x\to\infty}\frac{x}{1+x}=1$

and $\int_1^\infty \frac{1}{x^{3/2}}dx$ is convergent. Thus $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}dx$ is convergent.

(***iii***) Let $f(x)=\frac{1}{x^{1/3}(1+x)^{1/2}}$

we take $g(x)=\frac{1}{x^{1/3}\cdot x^{1/2}}=\frac{1}{x^{5/6}}$

We have $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=1$ and $\int_1^\infty \frac{1}{x^{5/6}}dx$ is convergent $\therefore \int_1^\infty f(x)\,dx$ is convergent.

➢ *Question*

Show that $\int_{-\infty}^\infty \frac{1}{1+x^2}dx$ is convergent.

Solution

We have

$$\int_{-\infty}^\infty \frac{1}{1+x^2}dx=\lim_{a\to\infty}\left[\int_{-a}^0 \frac{1}{1+x^2}dx+\int_0^a \frac{1}{1+x^2}dx\right]$$

$$=\lim_{a\to\infty}\left[\int_0^a \frac{1}{1+x^2}dx+\int_0^a \frac{1}{1+x^2}dx\right]=2\lim_{a\to\infty}\left[\int_0^a \frac{1}{1+x^2}dx\right]$$

$$=2\lim_{a\to\infty}\left|\tan^{-1}x\right|_0^a=2\left(\frac{\pi}{2}\right)=\pi$$

therefore the integral is convergent.

➢ *Question*

Show that $\int_0^\infty \frac{\tan^{-1}x}{1+x^2}dx$ is convergent.

Solution

$\because (1+x^2)\cdot\frac{\tan^{-1}x}{(1+x^2)}=\tan^{-1}x\to\frac{\pi}{2}$ as $x\to\infty$

Here $f(x)=\frac{\tan^{-1}x}{1+x^2}$ and $g(x)=1+x^2$

$\int_0^\infty \frac{\tan^{-1}x}{1+x^2}dx$ & $\int_0^\infty \frac{1}{1+x^2}dx$ behave alike.

$\because \int_0^\infty \frac{1}{1+x^2}dx$ is convergent $\therefore$ A given integral is convergent.

➢ *Question*

Show that $\int_0^{\infty} \frac{\sin x}{(1+x)^{\alpha}} dx$ converges for $\alpha > 0$.

Solution

$\int_0^{\infty} \sin x \, dx$ is bounded because $\int_0^{X} \sin x \, dx \le 2 \quad \forall \ x > 0$.

Furthermore the function $\frac{1}{(1+x)^{\alpha}}$, $\alpha > 0$ is monotonic on $[0, +\infty)$.

$\Rightarrow$ the integral $\int_0^{\infty} \frac{\sin x}{(1+x)^{\alpha}} dx$ is convergent.

➢ *Question*

Show that $\int_0^{\infty} e^{-x} \cos x \, dx$ is absolutely convergent.

Solution

$\because \ \left| e^{-x} \cos x \right| < e^{-x}$ and $\int_0^{\infty} e^{-x} \, dx = 1$

$\therefore$ the given integral is absolutely convergent. (comparison test)

➢ *Question*

Show that $\int_0^{1} \frac{e^{-x}}{\sqrt{1-x^4}} dx$ is convergent.

Solution

$\because \ e^{-x} < 1$ and $1 + x^2 > 1$

$$\therefore \ \frac{e^{-x}}{\sqrt{1-x^4}} < \frac{1}{\sqrt{(1-x^2)(1+x^2)}} < \frac{1}{\sqrt{1-x^2}}$$

Also
$$\int_0^{1} \frac{1}{\sqrt{1-x^2}} dx = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}} dx$$
$$= \lim_{\varepsilon \to 0} \sin^{-1}(1-\varepsilon) = \frac{\pi}{2}$$

$\Rightarrow \int_0^{1} \frac{e^{-x}}{\sqrt{1-x^4}} dx$ is convergent. (by comparison test)