Mathematical Methods of Physics

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Introduction to Mathematical Physics

Mathematical physics refers to the development of mathematical methods for application to problems Physics. It defines the field as "the application of mathematics to problems in physics and the development of mathematical methods suitable for such applications and for the formulation of physical theories.

Why to study Mathematical Physics

Physics is so interlinked with math that beyond the most elementary level of physics, you NEED to know math to do physics. ... Physics is a science. It uses mathematics as the logical framework to test ideas about how physical phenomena behave.
Mathematics is the primary tool used by physicists to construct models, make measurements and make predictions. Frequently, physicists also get degrees in mathematics to help them better understand the subtleties that would otherwise remain hidden in advanced physics.
“Don’t worry about your difficulties in mathematics. I assure you mine are greater.” Albert Einstein
This quote attributed to Albert Einstein rarely fails to raise a smile. It is reassuring for those who struggle with mathematics, but it is also surprising. Even a cursory brush with physics confirms that this science is highly mathematical. Whether it is describing the force of gravity, as was Einstein’s goal, understanding a simple light switch, or designing sophisticated GPS satellites, physicists need numbers to measure and to quantify, and they need mathematical equations to describe the relationships between physical objects and the forces that act on them. Mathematics is the indispensable language of physics — which is why Einstein’s admission strikes many as amusing

Mathematics does not just provide tools for physics. It can also drive physical insight. A striking example is Paul Dirac’s prediction of antimatter. Dirac was searching for an equation to describe the behaviour of electrons, tiny building blocks of matter, taking account of insights from special relativity and the other great success story of early twentieth-century physics, quantum mechanics. When he found a suitable mathematical expression he realised it contained twice as many pieces of information as necessary, leading him to suggest that each electron comes with an anti-particle,
the positron. Its existence, and that of many other anti-particles, was later confirmed in the laboratory.
Dirac’s efforts contributed vital pieces to a jigsaw puzzle physicists are still trying to complete today, aiming to describe all the fundamental forces and particles of nature

in one theoretical framework, known as the Standard Model of particle physics. The language of this framework is mathematics, and mathematical considerations have led to the discovery of other particles.

Differential Equations

An equation that involves one or more derivatives of an unknown function is called

a **differential equation**.

Order and Degree of a Differential Equation

The order of a differential equation is the order of the highest order derivative involved in the differential equation. The degree of a differential equation is the exponent of the highest order derivative involved in the differential equation when the differential equation satisfies the following conditions –

- All of the derivatives in the equation are free from fractional powers, positive as well as negative if any.
- There is no involvement of the derivatives in any fraction.
- There shouldn't be involvement of highest order derivative as a transcendental function,trigonometric or exponential, etc. The coefficient of any term containing the highest order derivative should just be a function of x, y, or some lower order derivative with respect to its independent variables.

Solved Examples

For a differential equation represented by a function f(x, y, y') = 0; the first order derivative is the highest order derivative that has involvement in the equation. Thus, the Order of such a Differential Equation = 1. In a similar way, work out the examples below to understand the concept better –

- $x\frac{d^2y}{dx^2}+y\frac{dy}{dx}+4y^2=1$: Order = 2
- $\sin\left(\frac{d^3y}{dx^3}\right)=\frac{dy}{dx}$: Order = 3
- For $\sqrt{(\frac{dy}{dx})^2+3y}=\frac{d^2y}{dx^2}$: Order = 2

Find the Degrees.

- $3y^2(\frac{dy}{dx})^3 - \frac{d^2y}{dx^2} = \sin(x^2)$

The highest order derivative involved here is of order 2, and its power = 1 in the equation. Thus, the order of the differential equation = 2, degree = 1.

- $\sqrt{1 + (\frac{dy}{dx})^2 = y\frac{d^3y}{dx^3}}$

  Since this equation involves fractional powers, we must first get rid of them. On squaring the equation, we get – $1 + (\frac{dy}{dx})^2 = y^2(\frac{d^3y}{dx^3})^2$. Now, we can clearly make out that the highest order derivative is of order 3 here i.e. order of the differential equation = 3 and since its power is 2 in the equation – the degree of the differential equation = 2.

- $\sin\left(\frac{dy}{dx}\right) + \frac{d^2y}{dx^2} + 3x = 0$
  Here, the highest order derivative is of order 2, and it has no involvement in any function. So, the order of the differential equation = 2, and degree = 1.

Variable Separable Equations

Simply put, a differential equation is said to be **separable** if the variables can be separated. That is, a separable equation is one that can be written in the form

$$F(y)\,dy = G(x)\,dx$$

Once this is done, all that is needed to solve the equation is to integrate both sides. The method for solving separable equations can therefore be summarized as follows:

Example 1: Solve the equation $2ydy = (x^2 + 1)dx$

Since this equation is already expressed in "separated" form, just integrate:

$$2y\,dy = (x^2 + 1)\,dx$$

$$\int 2y\,dy = \int (x^2 + 1)\,dx$$

$$y^2 = \tfrac{1}{3}x^3 + x + c$$

Example 2: Solve the equation

$$x\,dx + \sec x \sin y\,dy = 0$$

This equation is separable, since the variables can be separated:

$$\sec x \sin y \, dy = -x \, dx$$
$$\sin y \, dy = -x \cos x \, dx$$

The integral of the left-hand side of this last equation is simply

$$\int \sin y \, dy = -\cos y$$

and the integral of the right-hand side is evaluated using integration by parts:

$$\begin{aligned}\int -x \cos x \, dx &= -\int x \cos x \, dx \\ &= -[x \sin x - \int \sin x \, dx] \\ &= -(x \sin x + \cos x + c)\end{aligned}$$

The solution of the differential equation is therefore

$$\cos y = x \sin x + \cos x + c$$

Example 4: Find all solutions of the differential equation

$$(x^2 - 1) y^3 \, dx + x^2 \, dy = 0.$$

Separating the variables and then integrating both sides gives

$$x^2 \, dy = -(x^2 - 1) y^3 \, dx$$
$$\frac{dy}{y^3} = \frac{1 - x^2}{x^2} dx \qquad (\dagger)$$
$$\int y^{-3} dy = \int (x^{-2} - 1) \, dx$$
$$-\tfrac{1}{2} y^{-2} = -x^{-1} - x - c$$
$$\frac{1}{2y^2} = \frac{1}{x} + x + c$$

For More Practice:

If a differential equation can be written in the form

$$f(y)\,dy = \phi(x)\,dx$$

We say that variables are separable, y on left hand side and x on right hand side. We get the solution by integrating both sides

Working Rule:

Step 1. Separate the variables as $f(y)\,dy = \phi(x)\,dx$

Step 2. Integrate both sides as $\int f(y)\,dy = \int \phi(x)\,dx$

Step 3. Add an arbitrary constant C on R.H.S.

Example 4. *Solve* : $\dfrac{dy}{dx} = \dfrac{x(2\log x+1)}{\sin y + y\cos y}$. *(U.P.B. Pharm (C.O.) 2005)*

Solution. We have, $\dfrac{dy}{dx} = \dfrac{x(2\log x+1)}{\sin y + y\cos y}$

Separating the variables, we get

$$(\sin y + y\cos y)\,dy = \{x\,(2\log x + 1)\}\,dx$$

Integrating both the sides, we get $\int(\sin y + y\cos y)dy = \int\{x(2\log x+1)\}\,dx + C$

$$-\cos y + y\sin y - \int(1)\cdot\sin y\,dy = 2\int \log x\cdot x\,dx + \int x\,dx + C$$

$$\Rightarrow \quad -\cos y + y\sin y + \cos y = 2\left[\log x\cdot\frac{x^2}{2} - \int\frac{1}{x}\cdot\frac{x^2}{2}\,dx\right] + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = 2\log x\cdot\frac{x^2}{2} - \int x\,dx + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = 2\log x\cdot\frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = x^2\log x + C.$$ Ans.

Example 5. *Solve the differential equation.*

$$x^4\frac{dy}{dx} + x^3 y = -\sec(xy).$$ *(A.M.I.E.T.E., Winter 2003)*

Solution. $x^4\dfrac{dy}{dx} + x^3 y = -\sec(xy) \quad \Rightarrow \quad x^3\left(x\dfrac{dy}{dx} + y\right) = -\sec xy$

Put $v = xy$, $\dfrac{dv}{dx} = x\dfrac{dy}{dx} + y \quad \Rightarrow \quad x^3\dfrac{dv}{dx} = -\sec v$

$$\Rightarrow \quad \frac{dv}{\sec v} = -\frac{dx}{x^3} \quad \Rightarrow \int\cos v\,dv = -\int\frac{dx}{x^3} + c$$

$$\Rightarrow \sin v = \frac{1}{2x^2} + c \quad \Rightarrow \sin xy = \frac{1}{2x^2} + c$$ Ans.

Example 6. *Solve* : $\cos(x+y)dy = dx$

Solution. $\cos(x+y)\,dy = dx \quad \Rightarrow \quad \dfrac{dy}{dx} = \sec(x+y)$

On putting $x + y = z$

So that $1+\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{dy}{dx}=\frac{dz}{dx}-1$

$\frac{dz}{dx}-1=\sec z \Rightarrow \frac{dz}{dx}=1+\sec z$

Separating the variables, we get $\frac{dz}{1+\sec z}=dx$

On integrating, $\int\frac{\cos z}{\cos z+1}dz=\int dx \Rightarrow \int\left[1-\frac{1}{\cos z+1}\right]dz=x+C$

$$\int\left(1-\frac{1}{2\cos^2\frac{z}{2}-1+1}\right)dz=x+C$$

$$\int\left(1-\frac{1}{2}\sec^2\frac{z}{2}\right)dz=x+C \Rightarrow z-\tan\frac{z}{2}=x+C$$

$$x+y-\tan\frac{x+y}{2}=x+C$$

$$y-\tan\frac{x+y}{2}=C$$

Example 7. *Solve the equation.*

$(2x^2 + 3y^2 - 7)\,x\,dx - (3x^2 + 2y^2 - 8)\,y\,dy = 0$ *(U.P. II Semester, Summer 20*

Solution. We have

$$(2x^2 + 3y^2 - 7)\,x\,dx - (3x^2 + 2y^2 - 8)\,y\,dy = 0$$

Re-arranging (1), we get $\frac{x\,dx}{y\,dy}=\frac{3x^2+2y^2-8}{2x^2+3y^2-7}$

Applying componendo and dividendo rule, we get

$$\frac{x\,dx+y\,dy}{x\,dx-y\,dy}=\frac{5x^2+5y^2-15}{x^2-y^2-1} \Rightarrow \frac{x\,dx+y\,dy}{x^2+y^2-3}=5\left(\frac{x\,dx-y\,dy}{x^2-y^2-1}\right)$$

Multiplying by 2 both the sides, we get

$$\Rightarrow \left(\frac{2x\,dx+2y\,dy}{x^2+y^2-3}\right)=5\left(\frac{2x\,dx-2y\,dy}{x^2-y^2-1}\right)$$

Integrating both sides, we get

$$\log(x^2+y^2-3)=5\log(x^2-y^2-1)+\log C$$

$$\Rightarrow \quad x^2+y^2-3=C(x^2-y^2-1)^5$$

where C is arbitrary constant of integration.

EXERCISE 12.2

Solve the following differential equations :

1. $\frac{dx}{x}=\tan y\cdot dy$ **Ans.** $x\cos y=C$
2. $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ **Ans.** $\sin^{-1}y=\sin^{-1}x+C$
3. $y(1+x^2)^{1/2}dy+x\sqrt{1+y^2}\,dx=0$ **Ans.** $\sqrt{1+y^2}+\sqrt{1+x^2}=C$
4. $\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy=0$ **Ans.** $\tan x\tan y=C$

5. $(1+x^2)\,dy - x\,y\,dx = 0$ **Ans.** $y^2 = C(1+x^2)$
6. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$ **Ans.** $(e^y+1)\sin x = C$
7. $3\,e^x \tan y\,dx + (1-e^x)\,sec^2 y\,dy = 0$ **Ans.** $(1-e^x)^3 = C\tan y$
8. $(e^y + 2)\sin x\,dx - e^y \cos x\,dy = 0$ **Ans.** $(e^y+2)\cos x = C$
9. $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ **Ans.** $e^y = e^x + \frac{x^3}{3} + C$
10. $\frac{dy}{dx} = 1 + \tan(y-x)$ [Put $y - x = z$] **Ans.** $\sin(y-x) = e^{x+c}$
11. $(4x+y)^2 \frac{dx}{dy} = 1$ **Ans.** $\tan^{-1}\frac{4x+y}{2} = 2x + C$
12. $\frac{dy}{dx} = (4x+y+1)^2$ [**Hint.** Put $4x + y + 1 = z$] **Ans.** $\tan^{-1}\frac{4x+y+1}{2} = 2x + C$

Homogeneous Equations

A function $f(x,y)$ is said to be **homogeneous of degree n** if the equation

$$f(zx, zy) = z^n f(x,y)$$

holds for all $(x, y,)$, and z (for which both sides are defined).

Example 1: The function $f(x,y) = x^2 + y^2$ is homogeneous of degree 2, since

$$f(zx, zy) = (zx)^2 + (zy)^2$$

$$= z^2(x^2 + y^2) = z^2 f(x,y)$$

Example 2: The function $f(x,y) = \sqrt{x^8 - 3x^2}y^6$ is homogeneous of degree 4, since

$$f(zx, zy) = \sqrt{(zx)^8 - 3(zx)^2(zy)^6} = \sqrt{z^8(x^8 - 3x^2y^6)}$$

$$= \sqrt{z^8}\sqrt{x^8 - 3x^2y^6} = z^4 f(x,y)$$

Example 3: The function $f(x,y) = 2x + y$ is homogeneous of degree 1, since

$$f(zx, zy) = z(2x) + (zy)$$

$$f(zx, zy) = 2(zx) + (zy) = z(2x + y) = zf(x,y)$$

Example 4: The function $f(x,y) = x^3 - y^2$ is not homogeneous, since

$$f(zx, zy) = (zx)^3 - (zy)^2 = z^3x^3 - z^2y^2$$

which does not equal $z^n f(x,y) =$ for any n.

12.8 HOMOGENEOUS DIFFERENTIAL EQUATIONS

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$

is called a homogeneous equation if each term of $f(x, y)$ and $\phi(x, y)$ is of the same degree *i.e.*,

$$\frac{dy}{dx} = \frac{3xy + y^2}{3x^2 + xy}$$

In such case we put $\boxed{y = vx \text{ and } \frac{dy}{dx} = v + x\frac{dv}{dx}}$

The reduced equation involves v and x only. This new differential equation can be solved by *variables separable* method.

Working Rule

Step 1. Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ **Step 2.** Separate the variables.

Step 3. Integrate both the sides. **Step 4.** Put $v = \frac{y}{x}$ and simplify.

Example 8. *Solve the following differential equation*

$$(2xy + x^2)\frac{dy}{dx} = 3y^2 + 2xy$$ *(A.M.I.E.T.E. Dec. 2006)*

Solution. We have, $(2xy + x^2)\frac{dy}{dx} = 3y^2 + 2xy \Rightarrow \frac{dy}{dx} = \frac{3y^2 + 2xy}{2xy + x^2}$

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

On substituting, the given equation becomes $v + x\frac{dv}{dx} = \frac{3v^2x^2 + 2vx^2}{2vx^2 + x^2} = \frac{3v^2 + 2v}{2v + 1}$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{3v^2 + 2v - 2v^2 - v}{2v + 1} \qquad \Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 + v}{2v + 1} \Rightarrow \left(\frac{2v+1}{v^2+v}\right)dv = \frac{dx}{x}$$

$$\Rightarrow \quad \int\left(\frac{2v+1}{v^2+v}\right)dv = \int\frac{dx}{x} \qquad \Rightarrow \log(v^2 + v) = \log x + \log c$$

$$\Rightarrow \quad v^2 + v = cx \qquad \Rightarrow \quad \frac{y^2}{x^2} + \frac{y}{x} = cx$$

$$\Rightarrow \quad y^2 + xy = cx^3$$

Example 9. *Solve the equation :*

$$\frac{dy}{dx} = \frac{y}{x} + x\sin\frac{y}{x}$$

Solution. $$\frac{dy}{dx} = \frac{y}{x} + x\sin\frac{y}{x}$$

Put $y = vx$ in (1) so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = v + x\sin v$$

$$\Rightarrow \quad x\frac{dv}{dx} = x\sin v \quad \Rightarrow \quad \frac{dv}{dx} = \sin v$$

Separating the variable, we get

$$\Rightarrow \quad \frac{dv}{\sin v} = dx \quad \Rightarrow \quad \int \operatorname{cosec} v\, dv = \int dx + C$$

$$\log\tan\frac{v}{2} = x + C \quad \Rightarrow \quad \log\tan\frac{y}{2x} = x + C$$

EXERCISE 12.3

Solve the following differential equations:

1. $(y^2 - xy)\,dx + x^2\,dy = 0$ **Ans.** $\frac{x}{y} = \log x + C$
2. $(x^2 - y^2)\,dx + 2xy\,dy = 0$ *(AMIETE, June 2009)* **Ans.** $x^2 + y^2 = ax$
3. $x(y - x)\frac{dy}{dx} = y(y + x)$. **Ans.** $\frac{y}{x} - \log xy = a$
4. $x(x - y)\,dy + y^2\,dx = 0$ *(U.P. B. Pharm (C.O.) 2005)* **Ans.** $y = x\log Cy$
5. $\frac{dy}{dx} + \frac{x - 2y}{2x - y} = 0$ **Ans.** $y - x = C(x + y)^3$
6. $\frac{dy}{dx} = \tan\frac{y}{x} + \frac{y}{x}$ **Ans.** $\sin\frac{y}{x} = Cx$
7. $\frac{dy}{dx} = \frac{3xy + y^2}{3x^2}$ **Ans.** $3x + y\log x + Cy = 0$
8. $\frac{dy}{dx} = \frac{x^2 - 2y^2}{2xy}$ **Ans.** $4y^2 - x^2 = \frac{C}{x}$
9. $(x^2 + y^2)\,dy = xy\,dx$ **Ans.** $-\frac{x^2}{2y^2} + \log y = C$
10. $x^2y\,dx - (x^3 + y^3)\,dy = 0$ **Ans.** $\frac{-x^3}{3y^3} + \log y = C$
11. $(y^2 + 2xy)\,dx + (2x^2 + 3xy)\,dy = 0$ *(AMIETE, Summer 2004)* **Ans.** $xy^2(x + y) = C$
12. $(2xy^2 - x^3)\,dy + (y^3 - 2yx^2)\,dx = 0$ **Ans.** $y^2(y^2 - x^2) = Cx^{-2}$
13. $(x^3 - 3xy^2)\,dx + (y^3 - 3x^2y)\,dy = 0,\ y(0) = 1$ **Ans.** $x^4 - 6x^2y^2 + y^4 = 1$
14. $2xy^2\,dy - (x^3 + 2y^3)\,dx = 0$ **Ans.** $2y^3 = 3x^3\log x + 3x^3$
15. $x\sin\frac{y}{x}dy = \left(y\sin\frac{y}{x} - x\right)dx$ **Ans.** $\cos\frac{y}{x} = \log x + C$
16. $\left\{x\cos\frac{y}{x} + y\sin\frac{y}{x}\right\}y - \left\{y\sin\frac{y}{x} - x\cos\frac{y}{x}\right\}x\frac{dy}{dx} = 0$ **Ans.** $xy\cos\frac{y}{x} = a$
17. $\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$ **Ans.** $y + \sqrt{x^2 + y^2} = Cx^2$

12.10 LINEAR DIFFERENTIAL EQUATIONS

A differential equation of the form

$$\boxed{\frac{dy}{dx} + Py = Q} \quad (1)$$

is called a linear differential equation, where P and Q, are functions of x (but not of y).

In such case, multiply both sides of (1) by $e^{\int P dx}$

$$e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int P dx} \quad (2)$$

The left hand side of (2) is

$$\frac{d}{dx}\left[y.e^{\int P dx}\right]$$

(2) becomes $$\frac{d}{dx}\left[y.e^{\int P dx}\right] = Q.e^{\int P dx}$$

Integrating both sides, we get

$$y.e^{\int P dx} = \int Q.e^{\int P dx} dx + C$$

This is the required solution.

Note, $e^{\int P dx}$ is called the integrating factor.

Solution is $$\boxed{y \times [I.F.] = \int Q\,[I.F.]\,dx + C}$$

Working Rule:

Step 1: Convert the given equation to the standard form of linear differential equation

$$\textbf{i.e. } \frac{dy}{dx} + Py = Q$$

step 2: Find the integrating factor i.e. I.F= $\int e^{Pdx}$

step 3: Then the solution is $y(I.F) = \int Q(I.F)dx + C$

Example 12. *Solve:* $(x+1)\frac{dy}{dx} - y = e^x(x+1)^2$ *(A.M.I.E.T.E., Summer 2002)*

Solution. $\frac{dy}{dx} - \frac{y}{x+1} = e^x(x+1)$

Integrating factor $= e^{-\int \frac{dx}{x+1}} = e^{-\log(x+1)} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$

The solution is $y.\frac{1}{x+1} = \int e^x.(x+1).\frac{1}{x+1}dx = \int e^x dx$

$\frac{y}{x+1} = e^x + C$ Ans.

Example 13. *Solve a differential equation*

$(x^3 - x)\frac{dy}{dx} - (3x^2 - 1)y = x^5 - 2x^3 + x.$ *(Nagpur University, Summer 2008)*

Solution. We have $(x^3 - x)\frac{dy}{dx} - (3x^2 - 1)y = x^5 - 2x^3 + x$

$\Rightarrow \quad \frac{dy}{dx} - \frac{3x^2 - 1}{x^3 - x}y = \frac{x^5 - 2x^3 + x}{x^3 - x} \quad \Rightarrow \quad \frac{dy}{dx} - \frac{3x^2 - 1}{x^3 - x}y = x^2 - 1$

I.F. $= e^{\int -\frac{3x^2-1}{x^3-x}dx} = e^{-\log(x^3-x)} = e^{\log(x^3-x)^{-1}} = \frac{1}{x^3 - x}$

Its solution is

$y(\text{I.F.}) = \int Q(I.F.)\,dx + C \quad \Rightarrow \quad y\left(\frac{1}{x^3 - x}\right) = \int \frac{x^2 - 1}{x^3 - x}dx + C$

$\Rightarrow \quad \frac{y}{x^3 - x} = \int \frac{x^2 - 1}{x(x^2 - 1)}dx + C \quad \Rightarrow \quad \frac{y}{x^3 - x} = \int \frac{1}{x}dx + C$

$\Rightarrow \quad \frac{y}{x^3 - x} = \log x + C \quad \Rightarrow \quad y = (x^3 - x)\log x + (x^3 - x)C$ Ans.

Example 14. *Solve* $\sin x\frac{dy}{dx} + 2y = \tan^3\left(\frac{x}{2}\right)$ *(Nagpur University, Summer 2004)*

Solution. Given equation : $\sin x\frac{dy}{dx} + 2y = \tan^3\frac{x}{2} \quad \Rightarrow \quad \frac{dy}{dx} + \frac{2}{\sin x}y = \frac{\tan^3\frac{x}{2}}{\sin x}$

This is linear form of $\frac{dy}{dx} + Py = Q$

$\therefore \quad P = \frac{2}{\sin x}$ and $Q = \frac{\tan^3\frac{x}{2}}{\sin x}$

$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int \frac{2}{\sin x}dx} = e^{2\int \operatorname{cosec} x\,dx} = e^{2\log\tan\frac{x}{2}} = \tan^2\frac{x}{2}$

$\therefore$ Solution is $y.(\text{I.F.}) = \int \text{I.F.}(Q\,dx) + C$

$$y\tan^2\frac{x}{2} = \int \tan^2\frac{x}{2}\cdot\frac{\tan^3\frac{x}{2}}{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}\,dx + C = \frac{1}{2}\int\frac{\tan^4\frac{x}{2}}{\cos^2\frac{x}{2}}\,dx + C$$

$$= \frac{1}{2}\int \tan^4\frac{x}{2}.\sec^2\frac{x}{2}\,dx + C \qquad \ldots (1)$$

Putting $\tan\frac{x}{2} = t$ so that $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$ on R.H.S. (1), we get

$$y.\tan^2\frac{x}{2} = \frac{1}{2}\int t^4(2dt) + C \quad \Rightarrow \quad y\tan^2\frac{x}{2} = \frac{t^5}{5} + C$$

$$y\tan^2\frac{x}{2} = \frac{\tan^5\frac{x}{2}}{5} + C \qquad \text{Ans.}$$

EXERCISE 12.5

Solve the following differential equations:

1. $\frac{dy}{dx} + \frac{1}{x}y = x^3 - 3$ **Ans.** $xy = \frac{x^5}{5} - \frac{3x^2}{2} + C$
2. $(2y - 3x)\,dx + x\,dy = 0$ **Ans.** $y\,x^2 = x^3 + C$
3. $\frac{dy}{dx} + y\cot x = \cos x$ **Ans.** $y\sin x = \frac{\sin^2 x}{2} + C$
4. $\frac{dy}{dx} + y\sec x = \tan x$ **Ans.** $y = \frac{C - x}{\sec x + \tan x} + 1$
5. $\cos^2 x\frac{dy}{dx} + y = \tan x$ **Ans.** $y = \tan x - 1 + Ce^{-\tan x}$
6. $(x+a)\frac{dy}{dx} - 3y = (x+a)^5$ **Ans.** $2y = (x + a)^5 + 2C\,(x + a)^3$
7. $x\cos x\frac{dy}{dx} + y(x\sin x + \cos x) = 1$ **Ans.** $x\,y = \sin x + C\cos x$
8. $x\log x\frac{dy}{dx} + y = 2\log x$ **Ans.** $y\log x = (\log x)^2 + C$
9. $x\frac{dy}{dx} + 2y = x^2\log x$ **Ans.** $yx^2 = \frac{x^4}{4}\log x - \frac{x^4}{16} + C$
10. $dr + (2r\cot\theta + \sin 2\theta)\,d\theta = 0$ **Ans.** $r\sin^2\theta = \frac{-\sin^4\theta}{2} + C$
11. $\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$ **Ans.** $y = \sin x - 1 + Ce^{-\sin x}$
12. $(1-x^2)\frac{dy}{dx} + 2xy = x(1-x^2)^{1/2}$ **Ans.** $y = \sqrt{1-x^2} + C(1-x^2)$
13. $\sec x\frac{dy}{dx} = y + \sin x$ *(A.M.I.E.T.E., Dec 2005)* **Ans.** $y = -\sin x - 1 + ce^{\sin x}$
14. $y' + y\tan x = \cos x, y10) = 0$ *(A.M.I.E.T.E., June 2006)* **Ans.** $x\cos x$
15. Solve $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$ *(AMIETE, Dec. 2009)* **Ans.** $x = -\tan^{-1} y - 1 + ce^{\tan^{-1} y}$
16. Find the value of α so that e^2 is an integrating factor of differential equation $x(1 - y)\,dx - dy = 0$. *(A.M.I.E.T.E., Summer 2005)* **Ans.** $\alpha = \frac{1}{2}$

Exact Differential Equations

An exact differential equation is formed by directly differentiating its primitive (solution) without any other process.

$Mdx + Ndy = 0$ is said to be an exact differential equation if it satisfies the following condition

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Where $\frac{\partial M}{\partial y}$ denotes the differential co-efficient of M with respect to y keeping x constant and $\frac{\partial N}{\partial x}$, the differential coefficient of N w.r.t x keeping y constant.

Method to Solve Exact Differential Equation

Step 1: Integrate M w.r.t x keeping y constant

Step 2: Integrate N w.r.t y only those terms of N which do not contain x.

Step 3: Result 1+Result 2 = Constant

Example 28: Solve

$$(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$$

Solution: Here $M = (5x^4 + 3x^2y^2 - 2xy^3)$, N=$x^3y - 3x^2y^2 - 5y^4$

$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$, $\frac{\partial N}{\partial x} = 6x^2y - 6xy^2$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

Now $\int Mdx + \int (terms\ of\ N\ not\ containing\ x)dy = C$

$$\int (5x^4 + 3x^2y^2 - 2xy^3)dx + \int (-5y^4)dy = C$$

$$x^5 + x^3y^2 - x^2y^3 - y^5 = C$$

Example 31. *Solve :* $x\,dx + y\,dy = \dfrac{a^2(x\,dy - y\,dx)}{x^2 + y^2}$ *(U.P. Second Semester Summer 2005)*

Solution. We have, $x\,dx + y\,dy = \dfrac{a^2(x\,dy - y\,dx)}{x^2 + y^2}$

$$\Rightarrow \quad \left(x + \frac{a^2 y}{x^2 + y^2}\right)dx + \left(y - \frac{(a^2 x)}{x^2 + y^2}\right)dy = 0$$

Here, $\quad M = x + \dfrac{a^2 y}{x^2 + y^2}, \qquad N = y - \dfrac{a^2 x}{x^2 + y^2}$

Now, $\quad \dfrac{\partial M}{\partial y} = \dfrac{a^2\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}, \quad \dfrac{\partial N}{\partial x} = \dfrac{a^2\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}$

Since, $\quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Therefore equation is exact. Hence,

$$\int \left(x + \frac{a^2 y}{x^2 + y^2}\right)dx + \int y\,dy = C$$

$$\Rightarrow \quad \frac{x^2}{2} + a^2 y.\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right) + \frac{y^2}{2} = C$$

$$\Rightarrow \quad \left(\frac{x^2 + y^2}{2}\right) + a^2 \tan^{-1}\left(\frac{x}{y}\right) = C$$

Ans.

EXERCISE 12.7

Solve the following differential equation (1 – 11).

1. $(x + y - 10)\,dx + (x - y - 2)\,dy = 0$ **Ans.** $\frac{x^2}{2} + xy - 10x - \frac{y^2}{2} - 2y = C$

2. $(y^2 - x^2)\,dx + 2x\,y\,dy = 0$ **Ans.** $\frac{x^3}{3} = x\,y^2 + C$

3. $\left(1 + 3e^{x/y}\right)dx + 3e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$ *(R.G.P.V. Bhopal, Winter 2010)* **Ans.** $x + 3y\,e^{x/y} = C$

4. $(2x - y)\,dx = (x - y)\,dy$ **Ans.** $xy = x^2 + \frac{y^2}{2} + C$

5. $(y \sec^2 x + \sec x \tan x)\,dx + (\tan x + 2y)\,dy = 0$ **Ans.** $y \tan x + \sec x + y^2 = C$

6. $(ax + hy + g)\,dx + (hx + by + f)\,dy = 0$ **Ans.** $ax^2 + 2h\,xy + by^2 + 2gx + 2fy + C = 0$

7. $(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$ **Ans.** $\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = C$

8. $(2xy + e^y)\,dx + (x^2 + xe^y)\,dy = 0$ **Ans.** $x^2y + xe^y = C$

9. $(x^2 + 2ye^{2x})\,dy + (2xy + 2y^2e^{2x})\,dx = 0$ **Ans.** $x^2y + y^2\,e^{2x} = C$

10. $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\,\sin y)\,dy = 0$ *(M.D.U., 2010)*

Ans. $y\,(x + \log x) + x \cos y = C$

11. $(x^3 - 3xy^2)\,dx + (y^3 - 3x^3y)\,dy = 0,\ y(0) = 1$ **Ans.** $x^4 - 6x^2y^2 + y^4 = 1$

12. The differential equation $M\,(x, y)\,dx + N\,(x, y)\,dy = 0$ is an exact differential equation if

(a) $\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = 0$ (b) $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$ (c) $\frac{\partial M}{\partial y} \times \frac{\partial N}{\partial x} = 1$ (d) None of the above

LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

13.1 LINEAR DIFFERENTIAL EQUATIONS

If the degree of the dependent variable and all derivatives is one, such differential equations are called *linear differential equations e.g.*

(1) $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=x^2+x+1$ (2) $2\frac{d^2x}{dt^2}-\frac{dx}{dt}-3x=f(t)$

13.2 NON LINEAR DIFFERENTIAL EQUATIONS

If the degree of the dependent variable and / or its derivatives are of greater than 1 such differential equations are called one-linear differential equations.

(1) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y^2=\sin x$ (2) $\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2+y^2=e^x$ (3) $\left(\frac{d^2x}{dt^2}\right)^2+\frac{dx}{dt}+x=f(t)$

The order of a differential equation is the highest order of the derivative involved. All the above differential equations are of second order.

Fourier and Laplace transforms are mathematical tools to solve the differential equations.

13.3 LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH CONSTANT COEFFICIENTS

The general form of the linear differential equation of second order is

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

where P and Q are constants and R is a function of x or constant.

Differential operator. Symbol D stands for the operation of differential *i.e.*,

$$Dy=\frac{dy}{dx},\quad D^2y=\frac{d^2y}{dx^2}.$$

$\frac{1}{D}$ stands for the operation of integration.

$\frac{1}{D^2}$ stands for the operation of integration twice.

$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ can be written in the operator form.

$D^2y+P\,Dy+Qy=R \quad\Rightarrow\quad (D^2+PD+Q)y=R$

13.4 DIMENSION OF SPACE OF SOLUTION

A differential equation is said to have dimension k if the differential equation has k linearly independent solutions, $y_1, y_2 \ldots\ldots y_k$,

For example; $y''-5y'+6y=0$ has two dimensional solution e^{2x}, e^{3x}

$y'''-6y''+11y'+6y=0$ has three dimensional independent solution e^x, e^{2x}, e^{3x}.

13.5 NON-HOMOGENEOUS

Consider the differential equation

$$y''+P(x)y'+Q(x)y=F(x) \qquad ...(1)$$

(1) is said to be non-homogeneous if R.H.S. of (1) *i.e.*, $F(x)\neq 0$

13.6 HOMOGENEOUS

When $F(x) = 0$

Then (1) is said to be complementary homogeneous linear differential equation of (1).

13.7 SUPER POSITION OR LINEARITY PRINCIPLE

OR

FUNDAMENTAL THEOREM FOR HOMOGENEOUS DIFFERENTIAL EQUATION

Theorem. If $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous differential equation

$$y''+Py'+Qy=0 \qquad ...(1)$$

then the general solution of (1) is given by

General solution $= y = A\,y_1 + B\,y_2$.

Where A and B are arbitrary constants.

Proof. Here we have $y = Ay_1 + By_2$ ...(2)

Differentiating (2), we get $y' = Ay_1' + By_2'$ ...(3)

$$y'' = Ay_1'' + By_2'' \qquad ...(4)$$

On putting the values of y, y', y'' from (2), (3) and (4) in (1), we get

$$Ay_1''+By_2''+P(Ay_1'+By_2')+Q[Ay_1+By_2]=0$$

$$\Rightarrow \quad [Ay_1''+PAy_1'+QAy_1]+[By_2''+P.B.y_2'+Q.B.y_2]=0$$

$$\Rightarrow \quad A[y_1''+Py_1'+Q\,y_1]+B[y_2''+Py_2'+Qy_2]=0 \qquad ...(5)$$

Since y_1 and y_2 are independent solution of (1), *i.e.*,

$$y_1''+Py_1'+Qy_1=0 \qquad \text{[Form (3)]}$$

and

$$y_2''+Py_2'+Qy_2=0 \qquad \text{[From (4)]}$$

Then (5) becomes $A\,(0) + B\,(0) = 0$

Thus (2) is the general solution of homogeneous differential equation (1).

13.4 DIMENSION OF SPACE OF SOLUTION

A differential equation is said to have dimension k if the differential equation has k linearly independent solutions, $y_1, y_2 \ldots\ldots y_k$,

For example; $y'' - 5y' + 6y = 0$ has two dimensional solution e^{2x}, e^{3x}

$y''' - 6y'' + 11y' + 6y = 0$ has three dimensional independent solution e^x, e^{2x}, e^{3x}.

13.5 NON-HOMOGENEOUS

Consider the differential equation

$$y'' + P(x)y' + Q(x)y = F(x) \qquad \ldots(1)$$

(1) is said to be non-homogeneous if R.H.S. of (1) *i.e.*, $F(x) \neq 0$

13.6 HOMOGENEOUS

When $F(x) = 0$

Then (1) is said to be complementary homogeneous linear differential equation of (1).

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Theorem. If $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous differential equation

$$y'' + Py' + Qy = 0 \qquad \ldots(1)$$

then the general solution of (1) is given by

General solution $= y = A\,y_1 + B\,y_2$.

Where A and B are arbitrary constants.

Proof. Here we have $y = Ay_1 + By_2$ ...(2)

Differentiating (2), we get $y' = Ay_1' + By_2'$ ...(3)

$$y'' = Ay_1'' + By_2'' \qquad \ldots(4)$$

On putting the values of y, y', y'' from (2), (3) and (4) in (1), we get

$$Ay_1'' + By_2'' + P(Ay_1' + By_2') + Q[Ay_1 + By_2] = 0$$

$$\Rightarrow \quad [Ay_1'' + PAy_1' + QAy_1] + [By_2'' + P.B.y_2' + Q.B.y_2] = 0$$

$$\Rightarrow \quad A[y_1'' + Py_1' + Q\,y_1] + B[y_2'' + Py_2' + Qy_2] = 0 \qquad \ldots(5)$$

Since y_1 and y_2 are independent solution of (1), *i.e.*,

$$y_1'' + Py_1' + Qy_1 = 0 \qquad \text{[Form (3)]}$$

and

$$y_2'' + Py_2' + Qy_2 = 0 \qquad \text{[From (4)]}$$

Then (5) becomes $A\,(0) + B\,(0) = 0$

Thus (2) is the general solution of homogeneous differential equation (1).

13.8 LINEAR INDEPENDENCE AND DEPENDENCE

Two solutions $y_1(x)$ and $y_2(x)$ are said to be linearly independent if

$$Ay_1(x) + By_2(x) \neq 0$$

A and B are not equal to zero.

13.9 WRONSKIAN

We know that

$$W(y_1, y_2, x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

(1) If $W(y_1, y_2, x) = 0$, then $y_1(x)$ and $y_2(x)$ are linearly dependent.

(2) If $W(y_1, y_2, x) \neq 0$, then $y_1(x)$, $y_2(x)$ are linearly independent.

13.10 EXISTENCE OF LINEARLY INDEPENDENCE

Consider the equation

$$y'' + P(x)y' + P(x)y = 0 \quad \ldots(1$$

By the fundamental theorem we can say that (1) has two solutions $y_1(x)$ and $y_2(x)$
By Abel formula, these two solutions are independent.
The initial conditions are

$$y_1(x_0) = 1, \; y_1'(x) = 0$$

$$y_2(x_0) = 0, \; y_2'(x_0) = 1$$

$$W(y_1, y_2, x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Since $W(y_1, y_2, x_0) = 1$, the above pair is independent.

Example 1. *Check whether the following functions are linearly independent or not:*

$e^x \cos x$, $e^x \sin x$. *(Delhi University, 201*

Solution. We have,

$$y_1 = e^x \cos x, y_2 = e^x \sin x$$

$\Rightarrow$ $$y_1' = e^x \cos x - e^x \sin x \text{ and } y_2' = e^x \sin x + e^x \cos x$$

$$W(y_1 y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix}$$

$$= e^{2x} \begin{vmatrix} \cos x & \sin x \\ \cos x - \sin x & \sin x + \cos x \end{vmatrix}$$

$$= e^{2x} (\sin x . \cos x + \cos^2 x - \sin x . \cos x + \sin^2 x)$$

$$= e^{2x} \neq 0$$

Hence, $e^x \cos x$ and $e^x \sin x$ are linearly independent. Ans

13.11 STRUCTURE THEOREM FOR

$$y'' + P(x)y' + Q(x)y = 0 \quad \ldots(1$$

We have seen that differential equation (1) has two solutions $y_1(x)$ and $y_2(x)$, secondly these solutions are linearly independent.

Differentiating (3), we get

$$\frac{dW(y_1, y_2, x)}{dx} = y_1(x)y_2''(x) + y_1'(x)y_2'(x) - y_2(x)y_1''(x) - y_2'(x)y_1'(x)$$

$$= y_1(x)y_2''(x) - y_2(x)y_1''(x) = -a(x)(y_1y_2' - y_1'y_2)$$

$$\frac{dW(y_1, y_2, x)}{dx} = -a(x)\, W(y_1, y_2, x)$$

$$\Rightarrow \qquad \frac{d}{dx} W(y_1, y_2, x) + a(x)\, W(y_1, y_2, x) = 0$$

Equation (4) is a linear differential equation of first order

$$\text{Its I.F.} = e^{\int_{x_0}^{x} a(x).dx}$$

Hence, the solution of (4) is

$$W(y_1, y_2, x)\; e^{\int a x dx} = \text{constant}$$

$$W(y_1, y_2, x)\; e^{\int a(x)dx} = W[y_1, y_2, x_0]$$

$$[W(y_1, y_2, x_0) = 1]$$

$$W(y_1, y_2, x) = W(y_1, y_2, x_0)\; e^{-\int_{x_0}^{x} a(x)dx}$$

Example 2. (*a*) *Find Wronskian determinant*

(*b*) *Verify that the solutions satisfy the differential equation*

(*c*) *Show by Wronskian test the solutions are independent*

$$y_1 = e^{2x}, y_2 = e^{3x} \text{ and } y'' - 5y' + 6y = 0$$

Solution (*a*) $y_1 = e^{2x}, y_2 = e^{3x}$ and $y_1' = 2e^{2x}, \; y_2' = 3e^{3x}$

$$W(y_1, y_2, x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} = e^{5x}$$

(*b*) Now $y'' - 5y' + 6y = 0$

If $y_1 = e^{2x}$, then (2) becomes $4\, e^{2x} - 5\,(2e^{2x}) + 6\, e^{2x} = 0$

$\Rightarrow$ $(4 - 10 + 6)\, e^{2x} = 0 \Rightarrow 0 = 0$

If $y_2 = e^{3x}$ then (2) becomes $9\, e^{3x} - 5\,(3\, e^{3x}) + 6\, e^{3x} = 0$

$\Rightarrow$ $(9 - 15 + 6)\, e^{3x} = 0 \Rightarrow 0 = 0$

Thus, y_1 and y_2 satisfy the differential equation (2)

$$W(y_1, y_2, x) = e^{5x} \neq 0$$

Hence y_1 and y_2 are linearly independent.

Example 3. *If* $y_1 = e^{-x}\cos x, \quad y_2 = e^{-x}\sin x$ *and* $\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 2y = 0$, *then*

(a) Calculate Wronskian determinant

(b) Verify that y_1 *and* y_2 *satisfy the given differential equation.*

(c) Apply wronskian test to check that y_1, y_2 *are linearly independent.*

Solution. (*a*) $y_1 = e^{-x}\cos x, \qquad y_2 = e^{-x}\sin x$

$$y_1' = -e^{-x}\cos x - e^{-x}\sin x, \qquad y_2' = -e^{-x}\sin x + e^{-x}\cos x$$

$$W(y_1, y_2, x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x}\cos x & e^{-x}\sin x \\ -e^{-x}\cos x - e^{-x}\sin x & -e^{-x}\sin x + e^{-x}\cos x \end{vmatrix}$$

$$= e^{-x}\cos x\,.(-e^{-x}\sin x + e^{-x}\cos x) - e^{-x}\sin x\,(-e^{-x}\cos x - e^{-x}\sin x)$$

$$= e^{-2x}(-\sin x\cos x + \cos^2 x) + e^{-2x}(\sin x\cos x + \sin^2 x)$$

$$= e^{-2x}(\sin^2 x + \cos^2 x) \qquad \ldots(1)$$

$$= e^{-2x}$$

(*b*) $y''(x) + 2y'(x) + 2y = 0$ ...(2)

$$y_1 = e^{-x}\cos x$$

Patting the values of y'', y' and y in (2), we get

$$\text{L.H.S. of (2)} = e^{-x}(2\sin x) - 2e^{-x}(\cos x + \sin x) + 2e^{-x}\cos x$$

$$= e^{-x}(2\sin x - 2\cos x - 2\sin x + 2\cos x) = 0$$

Thus $y_1 = e^{-x}\cos x$ satisfies (2).

Similarly $y_2 = e^{-x}\sin x$ satisfies (2).

(*c*) $W(y_1, y_2, x) = e^{-2x}$

$$\neq 0$$

Hence y_1 and y_2 are linearly independent. **Ans.**

27.1 INTRODUCATION

We have already studied how to find the solution of a differential equation with constant coefficients.

There are many differential equations with variable coefficients like Bessel's equation, Legender's equation, Hermite's equations, Laguerrer's differential equation whose solution are not the combination of elementary functions.

The solutions are infinite series. In this chapter we will solve the differential equations by power series method and Frobenius method (extended power series method).

27.2 POWER SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS

We know that the solution of the differential equation

$$\frac{d^2y}{dx^2} - y = 0$$

are $y = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $y = e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$

These are power series solution of the given differential equation.

Another example of the differential equation $\frac{d^2y}{dx^2} + y = 0$

is satisfied by the power series

$$y = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

This idea leads to the methods of obtaining the solution of a linear differential equation of second order in series form.

The solution of the differential equation will be a series of ascending powers of x, the infinite series solution obtained will have its own region of convergence or validity.

27.3 ANALYTIC FUNCTION

A function $f(x)$ which can be expended in Taylor's series on interval containing the point x_0. The series converges to $f(x)$ for all x in the interval of convergence.

27.4 ORDINARY POINT

Consider the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

where P, Q are polynomials in x.

$x = a$ is an ordinary point of the above equation if the denominators of P and Q do not va for $x = a$. *i.e.* $(P \neq \infty, Q \neq \infty)$

For example :

(*i*) $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0 \Rightarrow \frac{d^2y}{dx^2}+\frac{x}{1+x^2}\frac{dy}{dx}-\frac{1}{1+x^2}y=0$

Here, denominator of P and Q *i.e.*, $1 + x^2$ is not equal to zero at $x = 0$. Therefore $x = 0$ is ordinary point for this differential equation.

(*ii*) $x^2\frac{d^2y}{dx^2}+(2x^2-x)\frac{dy}{dx}+y=0 \Rightarrow \frac{d^2y}{dx^2}+\frac{2x^2-x}{x^2}\frac{dy}{dx}+\frac{1}{x^2}y=0$

Here, the denominator of P and Q *i.e.*, x^2 is equal to zero for $x = 0$ $(P = \infty, Q = \infty)$. So, $x = 0$ is ordinary point for this equation.

Note. In this section, we have to solve those differential equation whose ordinary point is at x

27.5 SOLUTION OF THE DIFFERENTIAL EQUATION WHEN $X = 0$ IS AN ORDINARY POINT *i.e.* WHEN P DOES NOT VANISH FOR $X = 0$.

(*i*) Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots + a_k x^k + \ldots. = \sum_{k=0}^{\infty} a_k x^k$ be the solution of the differential equation.

(*ii*) Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ etc.

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \ldots. + ka_k x^{k-1} + \ldots + \ldots = \sum_{k=1}^{\infty} ka_k x^{k-1}$$

$$\frac{d^2y}{dx^2} = 2a_2 + 2.3\ a_3x + \ldots.. + a_k\ k(k-1)\ x^{k-2} + \ldots. = \sum_{k=2}^{\infty} a_k \cdot k\,(k-1).$$

(*iii*) Substitute the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ etc. in the given differential equation.

(*iv*) Calculate a_0, a_1, a_2 .... coefficients of various powers of x by equating the coeffici zero.

(*v*) Substitute the values of a_0, a_1, a_2 ... in the differential equation to get the require solution.

27.6 WORKING RULE TO SOLVE A DIFFERENTIAL EQUATION IF $X = 0$ IS ORDINARY POINT OF THE EQUATION

Step 1. Assume the solution $y = \Sigma a_k x^k$ *i.e.*,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots$$

Step 2. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of step (1).

Step 3. Substitute the values of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation.

Step 4. Equate to zero the coefficients of various power of x, to find out a_2, a_3 ..... in ter and a_1.

Step 5. Equate to zero the coefficient of x^k to get a recurrence relation of a's.

Step 6. Substitute $k = 0, 1, 2, 3, \ldots.$ in the recurrence relations to get the values of a $a_n, \ldots\ldots$.

Step 7. Substitute the values of $a_2, a_3, \ldots.$ etc (obtained in step 6) in the solution (1) **When $x = 0$ is the ordinary point.**

Example 1. *Solve in series the equation* $\frac{d^2y}{dx^2} + x^2 y = 0.$

Solution. We have, $\frac{d^2y}{dx^2} + x^2 y = 0$ ... (1)

The denominator of P and Q is not zero so $x = 0$ is the ordinary point.

Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8a^8 + ... + a_nx^n + ...$ ... (2)

Here $P \neq \infty$, and $Q \neq \infty$ for $x = 0$. So, $x = 0$ is the ordinary point of the equation (1).

Then $\frac{dy}{dx} = a_1 + 2a_2 x + 3\ a_3 x^2 + 4\ a_4 x^3 + 5\ a_5 x^4 + 6\ a_6 x^5 + 7\ a_7 x^6 + 8\ a_8 x^7 + ...$

$\frac{d^2y}{dx^2} = 2a_2 + 2 \cdot 3\ a_3 x + 3 \cdot 4\ a_4 x^2 + 4 \cdot 5\ a_5 x^3 + 5 \cdot 6\ a_6 x^4 + 6 \cdot 7\ a_7 x^5 + 7 \cdot 8\ a_8 x^6 + ...$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in (1), we get

$$2\ a_2 + 2.3\ a_3 x + 3.4\ a_4 x^2 + 4.5\ a_5 x^3 + 5.6\ a_6 x^4 + 6.7\ a_7 x^5 + 7.8\ a_8 x^6 + ...$$
$$+ x^2 (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + ...) = 0$$
$$2\ a_2 + 6\ a_3 x + (a_0 + 12\ a_4) x^2 + (a_1 + 20 a_5) x^3 + (a_2 + 30\ a_6) x^4 + ...$$
$$+ [a_{n-2} + (n+2)(n+1)\ a_{n+2}] x^n + ... = 0$$

Equating to zero the coefficients of the various powers of x, we obtain

$$a_2 = 0, \quad a_3 = 0$$

$$a_0 + 12a_4 = 0 \quad i.e. \quad a_4 = -\frac{1}{12}a_0$$

$$a_1 + 20\ a_5 = 0 \quad i.e. \quad a_5 = -\frac{1}{20}a_1$$

$$a_2 + 30\ a_6 = 0 \quad i.e. \quad a_6 = -\frac{1}{30}\ a_2 = 0 \qquad (a_2 = 0)$$

and so on. In general

$$a_{n-2} + (n+2)(n+1)\ a_{n+2} = 0 \quad \Rightarrow \quad \boxed{a_{n+2} = -\frac{a_{n-2}}{(n+1)(n+2)}}$$

Putting $n = 5$, $\quad a_7 = -\frac{a_3}{6 \times 7} = 0 \qquad (a_3 = 0)$

Putting $n = 6$, $\quad a_8 = -\frac{a_4}{7 \times 8} = \frac{a_0}{12 \times 7 \times 8}$

Putting $n = 7$, $\quad a_9 = -\frac{a_5}{8 \times 9} = \frac{a_1}{20 \times 8 \times 9}$

Putting $n = 8$, $\quad a_{10} = -\frac{a_6}{9 \times 10} = 0, \qquad (a_6 = 0)$

Putting $n = 9$, $\quad a_{11} = -\frac{a_7}{11 \times 10} = 0, \qquad (a_7 = 0)$

Putting $n = 10$, $\quad a_{12} = -\frac{a_8}{12 \times 11} = -\frac{a_0}{12 \times 8 \times 7 \times 11 \times 12}$

Substituting these values in (2), we get

$$y = a_0 + a_1 x - \frac{1}{12}a_0 x^4 - \frac{a_1}{20}x^5 + \frac{a_0}{12 \times 7 \times 8}x^8 + \frac{a_1}{20 \times 8 \times 9}x^9 - \frac{a_0}{12 \times 8 \times 7 \times 11 \times 12}x^{12} + ...$$

$$y = a_0\left(1 - \frac{1}{12}x^4 + \frac{x^8}{12 \times 7 \times 8} - \frac{x^{12}}{12 \times 8 \times 7 \times 11 \times 12} + ...\right) + a_1\left(x - \frac{x^5}{20} + \frac{x^9}{20 \times 8 \times 9} - ...\right)$$ **Ans.**

Example 2. *Solve the following differential equation in series*

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+4y=0. \qquad (U.P.\ II\ Semester\ summer$$

Solution. We have,

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+4y=0 \Rightarrow \frac{d^2y}{dx^2}-\frac{x}{1-x^2}\frac{dy}{dx}+\frac{4}{1-x^2}y=0$$

Here, $P \neq \infty$ and $Q \neq \infty$ for $x = 0$. So, $x = 0$ is an ordinary point of the given equation.

Assume the solution

$$y = \Sigma a_k x^k$$

$$\Rightarrow \quad y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots\ldots$$

$$y' = \Sigma a_k (k)x^{k-1}$$

$$y'' = \Sigma a_k (k)(k-1)x^{k-2}$$

Putting these values in the given equation, we get

$$(1-x^2)\Sigma a_k (k)(k-1)x^{k-2} - x\Sigma a_k (k)x^{k-1} + 4\Sigma a_k x^k = 0$$

$$\Rightarrow \quad \Sigma a_k (k)(k-1)x^{k-2} - \Sigma a_k (k)(k-1)x^k - \Sigma a_k (k)x^k + 4\Sigma a_k x^k = 0$$

$$\Rightarrow \quad \Sigma a_k (k)(k-1)x^{k-2} - \Sigma a_k x^k \{k(k-1)+k-4\} = 0$$

$$\Rightarrow \quad \Sigma a_k k(k-1)x^{k-2} - \Sigma a_k x^k (k^2-4) = 0$$

Now, equating the coefficient of x^k equal to zero. By putting $k + 2$ for k in first summ and $k = k$ in second summation, we have

$$a_{k+2}(k+2)(k+1) - a_k(k^2-4) = 0$$

$$a_{k+2}(k+2)(k+1) = a_k(k^2-4)$$

$$a_{k+2} = \frac{k^2-4}{(k+2)(k+1)}a_k = \frac{k-2}{k+1}a_k \quad \Rightarrow \quad \boxed{a_{k+2} = \frac{k-2}{k+1}a_k}$$

If $k = 0$, $\quad a_2 = -2a_0$

If $k = 1$, $\quad a_3 = -\frac{1}{2}a_1$

If $k = 2$, $\quad a_4 = \frac{0}{3}a_2 = 0$

If $k = 3$, $\quad a_5 = \frac{1}{4}a_3 = \frac{1}{4}\left(-\frac{1}{2}\right)a_1 = -\frac{a_1}{8}$

If $k = 4$, $\quad a_6 = \frac{2}{5}a_4 = \frac{2}{5}\times 0 = 0$

and so on.

Substituting these values is (2), we get

$$y = a_0 + a_1x + (-2a_0)x^2 + \left(-\frac{1}{2}a_1\right)x^3 + 0x^4 + \left(\frac{-a_1}{8}\right)x^5 + 0.x^6 + \ldots\ldots$$

$$= a_0 + a_1x - 2a_0x^2 - \frac{1}{2}a_1x^3 - \frac{a_1}{8}x^5 + \ldots\ldots$$

$$= a_0(1-2x^2) + a_1x\left(1 - \frac{x^2}{2} - \frac{x^4}{8} + \ldots\ldots\right)$$

Example 3. *Find the power series solution of* $(1-x^2)\,y'' - 2xy' + 2y = 0$ *about* $x = 0$. *(A.M.I.E.T.E., Winter 2000)*

Solution. Here, we have

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \Rightarrow \frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{2}{1-x^2}y = 0$$

Here $P \neq \infty$ and $Q \neq \infty$ for $x = 0$. So $x = 0$ is an ordinary point.

Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$ be the required solution.

$$y = \sum_{k=0}^{\infty} a_k x^k$$

Then $$\frac{dy}{dx} = \sum_{k=1}^{\infty} a_k \cdot k\, x^{k-1}, \qquad \frac{d^2y}{dx^2} = \sum_{k=2}^{\infty} a_k \cdot k(k-1)x^{k-2}$$

Substituting the values of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$(1-x^2)\Sigma a_k\, k\cdot(k-1)x^{k-2} - 2x\Sigma a_k \cdot k\, x^{k-1} + 2\Sigma a_k x^k = 0$$

$$\Rightarrow \quad \Sigma a_k \cdot k\cdot(k-1)x^{k-2} - \Sigma a_k k(k-1)x^k - 2\Sigma a_k . k x^k + 2\Sigma a_k x^k = 0$$

$$\Rightarrow \quad \Sigma a_k \cdot k\cdot(k-1)x^{k-2} - \Sigma[k(k-1)+2k-2]a_k x^k = 0$$

$$\Rightarrow \quad \Sigma a_k \cdot k\cdot(k-1)x^{k-2} - \Sigma(k^2+k-2)a_k x^k = 0$$

where the first summation extends over all values of k from 2 to ∞ and the second from $k = 0$ to ∞.

Now equating the coefficient of x^k equal to zero, we have

$$\Rightarrow \quad (k+2)(k+1)a_{k+2} - (k^2+k-2)a_k = 0$$

$$\Rightarrow \quad a_{k+2} = \frac{k^2+k-2}{(k+2)(k+1)}a_k = \frac{(k+2)(k-1)}{(k+2)(k+1)}a_k$$

$$\Rightarrow \quad \boxed{a_{k+2} = \frac{k-1}{k+1}a_k}$$

For $k = 0$ $$a_2 = -a_0,\ a_3 = 0,\ a_4 = \frac{a_2}{3} = -\frac{a_0}{3},\ a_5 = \frac{2}{4}a_3 = 0$$

For $k = 4$ $$a_6 = \frac{3}{5}a_4 = \frac{3}{5}\left(-\frac{a_0}{3}\right) = -\frac{a_0}{5},\ a_7 = \frac{4}{6}a_5 = 0, \text{ etc.}$$

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \ldots$$

$$= a_0 + a_1x - a_0x^2 + 0 - \frac{a_0}{3}x^4 + 0 - \frac{a_0}{5}x^6 + 0 + \ldots$$

$$\Rightarrow \quad y = a_0\left[1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} + \ldots\right] + a_1x$$

Ans.

Example 4. *Solve the differential equation* $y'' + (x-1)^2 y' - 4(x-1)y = 0$ *in series about the ordinary point* $x = 1$.

Solution. Put $x = t + 1$ (or $x - 1 = t$)

$$\left(\because \frac{dt}{dx} = 1\right)$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}$$

$$\Rightarrow \quad \frac{d}{dx} \equiv \frac{d}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

$\therefore$ The given equation becomes,

$$\frac{d^2y}{dt^2} + t^2y' - 4t\,y = 0$$

Now, $t = 0$ is an ordinary point.

Assume the solution to be

$$y = a_0 + a_1t + a_2t^2 + a_3t^3 + \ldots + a_nt^n + \ldots$$

then $$y' = a_1 + 2a_2t + 3a_3t^2 + \ldots + na_nt^{n-1} + \ldots$$

and $$y'' = 2a_2 + 3.2.\,a_3t + \ldots + n(n-1)\,a_nt^{n-2} + \ldots$$

Substituting these values is equation (1), we get

$$[2a_2 + 3.2\,.\,a_3t + 4.3\,.\,a_4t^2 + \ldots + n(n-1)\,a_nt^{n-2} + \ldots]$$
$$+ t^2[a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + \ldots + na_nt^{n-1}$$
$$- 4t[a_0 + a_1t + a_2t^2 + a_3t^3 + \ldots + a_nt^n + \ldots] = 0$$

Coeffieient of $t^0 = 0$

$$\Rightarrow \quad 2a_2 = 0 \quad \Rightarrow \quad \boxed{a_2 = 0}$$

Coefficient of $t = 0$

$$\Rightarrow \quad 3.2.\,a_3 - 4a_0 = 0 \quad \Rightarrow \quad \boxed{a_3 = \frac{2a_0}{3}}$$

Coefficient of $t^2 = 0$

$$\Rightarrow \quad 4.3.\,a_4 + a_1 - 4a_1 = 0$$

$$\Rightarrow \quad 12a_4 = 3a_1 \quad \Rightarrow \quad \boxed{a_4 = \frac{a_1}{4}}$$

Coefficient of $t^3 = 0$

$$\Rightarrow \quad 5.4.\,a_5 + 2a_2 - 4a_2 = 0 \Rightarrow a_5 = \frac{1}{10}a_2 \quad \Rightarrow \quad \boxed{a_5 = 0}$$

Coefficient of $t^4 = 0$

$$\Rightarrow \quad 6.5.\,a_6 + 3a_3 - 4a_3 = 0$$

$$a_6 = \frac{a_3}{6.5} = \frac{2a_0}{6.5.3} \quad \Rightarrow \quad \boxed{a_6 = \frac{a_0}{45}}$$

Now, coefficient of $t^n = 0$

$$\Rightarrow (n+2)(n+1)\,a_{n+2} + (n-1)\,a_{n-1} - 4a_{n-1} = 0$$

$$\Rightarrow \quad a_{n+2} = -\frac{(n-5)}{(n+2)(n+1)}\,a_{n-1}$$

Putting $n = 5, 6, 7, 8, \ldots$, we get

$$a_7 = 0$$

$$a_8 = \frac{-1}{8.7}\,a_5 = 0$$

$$= a_0\left(1+\frac{2}{3}t^3+\frac{1}{45}t^6-\frac{1}{1620}t^9+\ldots\right)+a_1\left(t+\frac{t^4}{4}+\ldots\right)$$

$$\Rightarrow\quad y = a_0\left[1+\frac{2}{3}(x-1)^3+\frac{1}{45}(x-1)^6-\frac{1}{1620}(x-1)^9+\ldots\right]+a_1\left[(x-1)+\frac{(x-1)^4}{4}+\ldots\right]$$

Where a_0 and a_1 are constants. Ans.

Note. Example 4 is solved about the regular singular point $x = 1$. Now we will solve the problems about the regular singular point $x = 0$.

We can also find the solution about a point other than $x = 0$, say about $x = c$. In this case we have to find out the series solution of powers of $(x - c)$, and the series is valid (convergent) around the point $x = c$.

In this method first we shift the origin to the point $x = c$, by putting $x = t + c$. The differential equation so obtained is solved by the method already discussed.

Singular Point

17.7 SINGULAR POINTS ABOUT $x = a$

Definition. Consider the equation ... (1)

$$y'' + P(x)y' + Q(x)y = 0$$

We assume that functions P and Q are not analytic ($P = \infty$ or $Q = \infty$) at $x = a$, so that $x = a$ is not an ordinary point but a *singular point of* (1).

There are two types of singular points. (1) Regular singular point, (2) Irregular singular points.

1. Regular Singular Point:

If $(x - a)P$ and $(x - a)^2 Q$ are not infinite at $x = a$, then $x = a$ is a regular singular point.

2. Irregular Singular Point:

If $(x-a)P$ and $(x-a)^2 Q$ are infinite at $x=a$, then $x=a$ is an irregular singular point.

27.8 SINGULAR POINT ABOUT $x = 0$.

Consider the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

If $(x-0)P$ and $(x-0)^2 Q$ are not infinite at $x=0$, then $x=0$ is a regular singular point. Otherwise it is an irregular singular point.

Note: In this section we will solve those differential equation where x_0 is a regular singular point.

Example 5. *Find regular singular points of the differential equation.*

$$2x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (x^2-4)y = 0$$

Solution. We have,

$$\frac{d^2y}{dx^2} + \frac{3}{2x}\frac{dy}{dx} + \frac{x^2-4}{2x^2}y = 0$$

$$P = \frac{3}{2x} \quad \text{and} \quad Q = \frac{x^2-4}{2x^2}$$

P and Q are not analytic (infinity) at $x=0$. So, $x=0$ is not ordinary point but as $(x-0)P$ and $(x-0)^2 Q$ are analytic (not infinite) so $x=0$ is a regular singular point.

$$x\cdot P = x\left(\frac{3}{2x}\right) = \frac{3}{2} \neq \infty \text{ at } x=0.$$

$$x^2Q = x^2\cdot\frac{x^2-4}{2x^2} = \frac{1}{2}(x^2-4) \neq \infty \text{ at } x=0$$

Example 6. *Find regular singular points of the differential equation.*

$$x(x-2)^2y'' + 2(x-2)y' + (x+3)y = 0$$

Solution. Here, we have

$$P = \frac{2(x-2)}{x(x-2)^2} = \frac{2}{x(x-2)} \quad \text{and} \quad Q = \frac{x+3}{x(x-2)^2}$$

P and Q are not analytic ($P=\infty$, $Q=\infty$) at $x=0$ and $x=2$.

Hence both these points are singular points of (1).

(*i*) At $x=0$

$$xP = x\cdot\frac{2}{x(x-2)} = \frac{2}{x-2} \neq \infty \quad \text{at } x=0$$

$$x^2Q = x^2\cdot\frac{x+3}{x(x-2)^2} = \frac{x(x+3)}{(x-2)^2} \neq \infty \quad \text{at } x=0$$

Hence, xP and x^2Q are analytic ($xP \neq \infty$, $x^2P \neq \infty$) at $x=0$. So $x=0$ is a regular singular point.

(*ii*) At $x=2$

$$(x-2)P = (x-2)\cdot\frac{2}{x(x-2)} = \frac{2}{x} \neq \infty \text{ at } x=2$$

$$(x-2)^2Q = (x-2)^2\frac{(x+3)}{x(x-2)^2} = \frac{x+3}{x} \neq \infty \text{ at } x=2.$$

Since both $(x-2)P$ and $(x-2)^2Q$ are analytic ($(x-2)P \neq \infty$, $(x-2)^2Q \neq 0$) at $x=2$, $x=2$ is a regular singular point.

The solution of a differential equation about a regular singular point can be obtained.

The cases of irregular singular points are beyond the scope of this book.