Sets, Relations, and Languages

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SETS (1)

A set is a collection of objects.

For example,

L = {a, b, c, d}

The collection of the four letters *a, b, c, and d is a set, which we may name L;*

Objects comprising a set are called its ***elements*** or ***members***.

For example, ***b*** is an element of the set ***L****; in symbols,* $\boldsymbol{b \in L}$*. Sometimes we simply say that* ***b*** *is* in ***L****, or that* ***L*** *contains* ***b****.*

On the other hand, ***z*** *is not an element of* ***L****, and* we write $\boldsymbol{z \notin L}$*.*

SETS (2)

In a set we do not distinguish repetitions of the elements.

For example:

The set ***{red, blue, red}*** is the same set as ***{red, blue}***.

The order of the elements is immaterial.

For example:

{3, 1, 9}, ***{9, 3, 1}***, and ***{I, 3, 9}*** are the same set.

To summarize: Two sets are equal (that is, the same) if and only if they have the same elements.

Sets (3)

The elements of a set need not be related in any way (other than happening to be all ***members*** of the same set);

For example: ***{3, red, {d, blue}}*** *is a set with* three elements, one of which is itself a set.

Singleton set

A set having only one element is called a Singleton. For example:

{l} is the set with 1 as its only element; thus ***{1}*** and 1 are quite different.

Empty set

A set with no element at all. Naturally, there can be only one such set: it is called the empty set, and is denoted by $\emptyset$.

Any set other than the empty set is said to be non-empty.

SETS (4)

The set $\mathbb{N}$ of natural numbers is infinite;

we may suggest its elements by writing $\boldsymbol{\mathbb{N} = \{ 1, 2, 3, \ldots \}}$, *using the three dots* and your intuition in place of an infinitely long list. A set that is not infinite is finite.

The set of integers Z is written as:

$$Z = \{ \ldots , -2, -1, 0, 1, 2, \ldots \}$$

SETS (5)

Another way to specify a set is by referring to other sets and to properties that elements may or may not have. Thus if ***I = {l, 3, 9}*** and ***G = {3, 9}***, ***G*** may be described as the set of elements of ***I*** *that are greater than* **2**. *We write* this fact as follows.

G = { x | x Є I and x is greater than 2}

SETS (6)

In general, if a set ***A*** *has been defined and* ***P*** *is a property that elements of* ***A*** may or may not have, then we can define a new set

B = { x | x ∈ A and x has property P }

As another example, the set of odd natural numbers is

O = { x | x ∈ N and x is not divisible by 2 }

SETS (7)

A set ***A*** *is a subset of a set* ***B*** *--in symbols,* ***A*** ⊆ ***B*** *- if each element of A is also an element of B.*

Thus 0 ⊆ N, since each odd natural number is a natural number.

Note that any set is a subset of itself. If ***A*** *is a subset of* ***B*** but ***A*** *is not the same as* ***B****, we say that* ***A*** *is a proper subset of* ***B*** *and write* ***A*** ⊂ ***B****.*

The empty set is a subset of every set.

For if ***B*** *is any set,* then ***Ø*** ⊆ ***B****, since each element of* ***Ø*** *(of which there are none) is also an element* of ***B.***

SETS (8)

To prove that two sets ***A*** *and* ***B*** *are equal, we may prove that* ***A*** ⊆ ***B*** *and* ***B*** ⊆ ***A****. Every element of* ***A*** *must then be an element of* ***B*** *and vice versa, so* that ***A*** *and* ***B*** *have the same elements and* ***A = B****.*

SETS (9)

Two sets can be combined to form a third by ***set operations***, *just* as numbers are combined by arithmetic operations such as addition.

One set operation is union: the union of two sets is that set having as elements the objects that are elements of at least one of the two given sets, and possibly of both.

We use the symbol U to denote union, so that

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}.$$

For example,

$$\{l, 3, 9\} \cup \{3, 5, 7\} = \{l, 3, 5, 7, 9\}.$$

SETS (10)

The intersection of two sets is the collection of all elements the two sets have in common; that is,

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$

For example:

$\{I, 3, 9\} \cap \{3, 5, 7\} = \{3\}$, and

$\{1,3,9\} \cap \{a, b, c, d\} = \emptyset$

SETS (11)

The set $\mathbb{N}$ of natural numbers is infinite;

we may suggest its elements by writing ***$\mathbb{N}$ = { 1, 2, 3, … }***, *using the three dots* and your intuition in place of an infinitely long list. A set that is not infinite is finite.

The set of integers **Z** is written as:

Z = { … , -2, -1, 0, 1, 2, … }

Another way to specify a set is by referring to other sets and to properties that elements may or may not have. Thus if ***I = {l, 3, 9}*** *and* ***G = {3, 9}***, *G* may be described as the set of elements of *I that are greater than 2. We write* this fact as follows. ***G = {x : x Є I and x is greater than 2}.***

SETS (12)

The **difference of two sets *A and B, denoted by A - B, is the set of all*** elements of *A that are not elements of B.*

A - B = {x| x ∈ A and x ∉ B}.

For example,

{1, 3, 9} - {3, 5, 7} = {1, 9}.

SETS (13)

if *A, B, and C are sets, the following laws hold*

- ***Idempotency***

$$A \cup A = A$$
$$A \cap A = A$$

- ***Commutativity***

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- ***Associativity***

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

- ***Distributivity***

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

Sets (14)

Absorption

$$(A \cup B) \cap A = A$$

$$(A \cap B) \cup A = A$$

DeMorgan's laws

$$A - (B \cup C) = (A - B) \cap (A - C)$$

$$A - (B \cap C) = (A - B) \cup (A - C)$$

SETS (15)

<u>Example</u> : Let us prove the first of De Morgan's laws.
Let *L = A - (B U C)* and *R = (A - B) ∩ (A - C);*
we are to show that *L = R.*
We do this by showing (a) L ⊆ R and (b) R ⊆ L.

(a) Let *x be any element of L; then x E A, but x ∉ B and x ∉ C. Hence x is an element of both A - B and A - C, and is thus an element of R.*
Therefore *L* ⊆ *R.*

(b) Let *x* ∈ *R; then x is an element of both A - B and A - C, and is* therefore in *A but in neither B nor C. Hence x* ∈ *A but x* ∉ *B U C, so x* ∈ *L.* Therefore *R* ⊆ *L, and we have established that L = R.*

SETS (16)

Two sets are disjoint if they have no element in common, that is, if their intersection is empty.

RELATIONS

- Mathematics deals with statements about objects and the relations between them. It is natural to say,

For example, that "less than" is a relation between objects of a certain kind -namely, numbers - which holds between 4 and 7 but does not hold between 4 and 2, or between 4 and itself.

ORDERED PAIR

- We write the ordered pair of two objects ***a*** *and* ***b*** *as* ***(a, b)****;* ***a*** *and* ***b*** *are called* the components of the ordered pair ***(a, b)***. *The ordered pair* ***(a, b)*** *is not the* same as the set ***{a, b}***. *First, the order matters:* ***(a, b)*** *is different from* ***(b, a)****,* whereas ***{a, b} = {b, a}***.

Second, the two components of an ordered pair need not be distinct; ***(7, 7)*** is a valid ordered pair. Note that two ordered pairs ***(a, b)*** and ***(c, d)*** *are equal only when* ***a = c*** *and* ***b = d***.

CARTESIAN PRODUCT

- The Cartesian product of two sets ***A*** *and* ***B****, denoted by* ***A x B****, is the* set of all ordered pairs ***(a, b)*** *with* ***a Є A*** *and* ***b Є B****. For example,*

{1, 3, 9} x {b, c, d} = { (1, b), (1, c), (1, d), (3, b), (3 , c), (3, d), (9, b), (9 , c), (9, d) } .

BINARY RELATION

A binary relation on two sets ***A*** *and* ***B*** *is a subset of* ***A x B****.*

For example,

{ (1, b), (1, c), (3, d), (9, d) } *is a binary relation on* ***{1, 3, 9}*** *and* ***{b, c, d}****. And*

{(i, j) | i, j ∈ ℕ and i < j} *is the less-than relation; it is a subset of* ***ℕ x ℕ*** --- often the two sets related by a binary relation are identical.

FUNCTIONS AND RELATIONS

- Functions are central to mathematics.

A function is an object that sets up an input-output relationship. A function takes an input and produces an output.

- In every function, the same input always produces the same output. If ***f*** is a function whose output value is ***b*** when the input value is ***a***, we write

$$f(a) == b.$$

A function also is called a mapping, and, if ***f(a) == b***, we say that ***f*** maps ***a*** to ***b***.

EXAMPLES

- The absolute value function abs takes a number x as input and returns x if x is positive and -x if x is negative.

  Thus ***abs(2) == abs(-2) == 2.***

- Addition is another example of a function, written add. The input to the addition function is a pair of numbers, and the output is the sum of those numbers.

- The set of possible inputs to the function is called its **domain**. The outputs of a function come from a set called its **range**. The notation for saying that f is a function with domain D and range R is

$$\boldsymbol{f: D \longrightarrow R}$$

- In the case of the function abs, if we are working with integers, the domain and the range are Z, so we write

$$\boldsymbol{abs\text{: } Z \rightarrow Z}$$

- In the case of the addition function for integers, the domain is the set of pairs of integers Z x Z and the range is Z, so we write

$$\boldsymbol{add\text{: } Z \times Z \rightarrow Z}$$

EXAMPLE

- Consider the function

f: $\{0, 1, 2, 3, 4\} \to \{0, 1, 2, 3, 4\}$.

n	$f(n)$
0	1
1	2
2	3
3	4
4	0

This function adds 1 to its input and then outputs the result modulo 5.